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Integral Equations and Operator Theory

ON A NEW CLASS OF STRUCTURED MATRICES

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In this paper we continue the study of structured matrices which admit a linear complexity inversion algorithm. The new class which is studied here appears naturally as the class of matrices of input output operators for discrete time dependent descriptor linear systems. The algebra of such operators is analyzed. Multiplication and inversion algorithms of linear complexity are presented and their implementation is illustrated.

0. Introduction

In this paper we continue the study of structured matrices which admit a linear complexity inversion algorithm. Such algorithms exist for diagonal plus semiseparable matrices and band matrices. The new class which is studied here appears naturally as the class of matrices of input output operators for descriptor linear systems and contains both diagonal plus semiseparable and band matrices.

Let R be a square matrix of size $N \times N$. Let n be a number such that the entries of lower triangular part of the matrix R have the form

$$R_{ij} = p_i a^{\times}_{ij} q_j, \quad 1 \leq j < i \leq N, \tag{0.1}$$

where p_i are n-dimensional rows, q_j are n-dimensional columns, $a^{\times}_{ij} = a_{i-1} \cdots a_{j+1}$, $i > j+1$, $a^{\times}_{i+1,i} = I_n$, a_k are $n \times n$ matrices. The elements p_i $(i = 2, \dots, N)$, q_j $(j = 1, \dots, N-1)$, a_k $(k = 2, \dots, N-1)$ are called lower generators of the matrix R and the number n is called order of lower generators. Let n_1 be a minimal value of n for which (0.1) holds. Then the matrix R is called lower quasiseparable of order n_1. The definition of upper quasiseparable matrix and upper generators is similar. If a matrix R is lower quasiseparable of order n_1 and upper quasiseparable of order n_2 then it is called quasiseparable of order (n_1, n_2).

It is well known (see for instance [GL, p.92-95]) that for a band matrix R the solution of the system $Rx = y$ may be computed at the cost $O(N)$ arithmetic operations. As was shown for instance by Asplund in [A] inverse to a band matrix with nonzero entries on

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external diagonals belongs to the class of diagonal plus semiseparable matrices. Let us remind that a matrix is said to be semiseparable of order (n_1, n_2) if it is composed of the lower triangular part of some matrix of rank n_1 at most and of the upper triangular part of another matrix of rank n_2 at most. Probably the first time the linear complexity algorithm for inversion of diagonal plus semiseparable matrices was suggested by Gohberg, Kailath, Koltracht in [GKK1], [GKK2] in assumption that the matrix R is strongly regular, i.e. all its leading minors are non-vanishing. In [GKK1], [GKK2] it was established that lower triangular and upper triangular factors of LDU factorization of diagonal plus semiseparable matrix R are also diagonal plus semiseparable and moreover generators of these factors may be expressed via generators of original matrix using linear complexity by N algorithm. Then the solution of every corresponding triangular system may be computed in $O(N)$ operations. Another approach to inversion of diagonal plus semiseparable matrices was suggested by Gohberg and Kaashoek in [GK]. In [GK] such matrices arose as input-output ones for discrete linear systems with boundary conditions. In [GK] under the assumption that external coefficients of the system are nonvanishing an explicit inversion formula for the input output matrix was obtained. It was established by the authors in [EG2] that using the formula from [GK] one can obtain the solution of equation $Rx = y$ for $O(N)$ operations. This formula was analyzed in detail by the authors in [EG1], [EG2]. It turned out that one can obtain an equivalent representation of the entries of the inverse matrix which is valid without any limitations on the matrix except of invertibility, and moreover the relations obtained are a basis for linear complexity inversion algorithm. Analysis of representations obtained in [EG1], [EG2] showed that inverse to diagonal plus semiseparable matrix belongs in general to a wider class. This new class contains both diagonal plus semiseparable matrices and band matrices and is contained in the class of quasiseparable matrices. This is a second reason for our interest in this class.

The object of the paper is the detailed study of the properties of quasiseparable matrices. It turns out that similarly to a diagonal plus semiseparable matrix a quasiseparable matrix of general form may be treated as an input output one for discrete time varying linear system with boundary conditions. However it is necessary that a part of state space equations of the system is a forward recursion and another part is a backward recursion. Such systems are called descriptor systems. We consider in detail the algebraic properties of the class of quasiseparable matrices. As one of the results one can mention the property that the inverse to quasiseparable matrix is again a quasiseparable matrix (a result which does not hold for diagonal plus semiseparable and band matrices). Linear complexity by N multiplication and inversion algorithms are developed in the paper. The implementation of these algorithms is illustrated by results of numerical experiments.

The paper consists of 9 sections:

1. Definitions
2. Quasiseparable Matrices and Descriptor Systems

Note that inversion algorithms and their implementation for quasiseparable matrices of order (1,1) will be considered in more detail by the authors in a later paper.

1. Definitions

Let $\{a_k\}, k = 1,\dots,N$ be a family of square matrices of the same size. For positive integers i,j define the operation $a^{\times}_{ij}$ as follows: $a^{\times}_{ij} = a_{i-1}\cdots a_{j+1}$ for $N \geq i > j+1 \geq 2$, $a^{\times}_{ij} = a_{i+1}\cdots a_{j-1}$ for $N \geq j > i+1 \geq 2$, $a^{\times}_{k+1,k} = a^{\times}_{k,k+1} = I$ for $1 \leq k \leq N-1$, $a^{\times}_{k,k} = 0$ for $1 \leq k \leq N$.

We consider a class of matrices R for which either lower triangular part or upper triangular part or both of them has a special structure. Let R be a square matrix of size $N \times N$. Let n be a number such that entries of the lower triangular part of matrix R have the form

$$R_{ij} = p_i a^{\times}_{ij} q_j, \quad 1 \leq j < i \leq N, \tag{1.1}$$

where p_i are n-dimensional rows, q_j are n-dimensional columns, a_k are $n \times n$ matrices. The elements p_i $(i = 2,\dots,N)$, q_j $(j = 1,\dots,N-1)$, a_k $(k = 2,\dots,N-1)$ are called *lower generators* of the matrix R and the number n is called *order of lower generators.* Let n_1 be a minimal value of n for which (1.1) holds. Then the matrix R is called *lower quasiseparable of order* n_1.

Let n be a number such that entries of the upper triangular part of matrix R have the form

$$R_{ij} = g_i b^{\times}_{ij} h_j, \quad 1 \leq i < j \leq N, \tag{1.2}$$

where g_i are n-dimensional rows, h_j are n-dimensional columns, b_k are $n \times n$ matrices. The elements g_i $(i = 1,\dots,N-1)$, h_j $(j = 2,\dots,N)$, b_k $(k = 2,\dots,N-1)$ are called *upper generators* of the matrix R and the number n is called *order of upper generators.* Let n_2 be a minimal value of n for which (1.2) holds. Then the matrix R is called *upper quasiseparable of order* n_2.

If a matrix R of size $N \times N$ is lower quasiseparable of order n_1 and upper quasiseparable of order n_2 then it is called *quasiseparable of order* (n_1, n_2). More precisely quasiseparable of order (n_1, n_2) matrix is a matrix of the form

$$R_{ij} = \begin{cases} p_i a^{\times}_{ij} q_j, & 1 \leq j < i \leq N, \\ d_i, & 1 \leq i = j \leq N, \\ g_i b^{\times}_{ij} h_j, & 1 \leq i < j \leq N. \end{cases} \tag{1.3}$$

The elements p_i $(i = 2,\dots,N)$, q_j $(j = 1,\dots,N-1)$, a_k $(k = 2,\dots,N-1)$; g_i $(i = 1,\dots,N-1)$, h_j $(j = 2,\dots,N)$, b_k $(k = 2,\dots,N-1)$; d_k $(k = 1,\dots,N)$ are called *generators* of the matrix R.

The class under consideration is a generalization of two well-known classes of structured matrices: band matrices and diagonal plus semiseparable matrices. If in (1.3) $a_k = a$, $b_k = b$ $(k = 2,\dots,N-1)$ and $a^{n_1} = 0$, $b^{n_2} = 0$ then the matrix R is a band matrix. If $a_k = I_{n_1}$, $b_k = I_{n_2}$ $(k = 2,\dots,N-1)$ then we obtain a diagonal plus semiseparable matrix.

2. Quasiseparable Matrices and Descriptor Systems

Let us consider discrete time system of the following type:

$$\begin{cases} \chi_{k+1} = a_k\chi_k + q_k x_k, \quad k = 1,\dots,N-1, \\ \eta_{k-1} = b_k\eta_k + h_k x_k, \quad k = N,\dots,2, \\ y_k = p_k\chi_k + g_k\eta_k + d_k x_k, \quad k = 1,\dots N, \\ M_1\begin{pmatrix}\chi_1\\ \eta_1\end{pmatrix} + M_2\begin{pmatrix}\chi_N\\ \eta_N\end{pmatrix} = 0. \end{cases} \tag{2.1}$$

Here $x = (x_k)_{k=1}^N$ is the input of the system, $y = (y_k)_{k=1}^N$ is the output, χ_k and η_k are the state space variables of sizes n_1 and n_2 correspondingly; the coefficients are square matrices a_k, b_k of sizes n_1, n_2 correspondingly, vector columns q_k, h_k of sizes n_1, n_2 respectively, vector rows p_k, g_k of sizes n_1, n_2 respectively, numbers d_k. The boundary conditions are determined by two matrices M_1, M_2 of size $m \times m$, where $m = n_1 + n_2$. The number m is called the *order* of the system.

In addition to the matrices $a_{ij}^{\times}$, $b_{ij}^{\times}$ we use here the matrices $a_i^{\#} = a_{i1}^{\times}a_1$ for $N \geq i \geq 2$, $a_1^{\#} = I_{n_1}$; $b_i^{\#} = b_{iN}^{\times}b_N$ for $N-1 \geq i \geq 1$, $b_N^{\#} = I_{n_2}$.

The system (2.1) is said to have *well posed* boundary conditions if the homogeneous equation

$$\begin{cases} \chi_{k+1} = a_k\chi_k, \quad k = 1,\dots,N-1, \\ \eta_{k-1} = b_k\eta_k, \quad k = N,\dots,2, \\ M_1\begin{pmatrix}\chi_1\\ \eta_1\end{pmatrix} + M_2\begin{pmatrix}\chi_N\\ \eta_N\end{pmatrix} = 0 \end{cases} \tag{2.2}$$

has the trivial solution only. This happens if and only if $\det M \neq 0$, where

$$M = M_1\begin{pmatrix} I_{n_1} & 0 \\ 0 & b_1^{\#}\end{pmatrix} + M_2\begin{pmatrix} a_N^{\#} & 0 \\ 0 & I_{n_2}\end{pmatrix}. \tag{2.3}$$

Indeed the solution of (2.2) satisfies the relations

$$\chi_k = a_k^{\#}\chi_1, \; k = 1,\dots,N; \quad \eta_k = b_k^{\#}\eta_N, \; k = N,\dots,1. \tag{2.4}$$

In particular $\chi_N = a_N^{\#}\chi_1$, $\eta_1 = b_1^{\#}\eta_N$. The boundary conditions yield

$$M_1 \begin{pmatrix} \chi_1 \\ b_1^{\#}\eta_N \end{pmatrix} + M_2 \begin{pmatrix} a_N^{\#}\chi_1 \\ \eta_N \end{pmatrix} = M \begin{pmatrix} \chi_1 \\ \eta_N \end{pmatrix} = 0. \tag{2.5}$$

If $\det M \neq 0$ then $\chi_1 = 0$, $\eta_N = 0$ and by virtue of (2.4) the equation (2.2) has the trivial solution only. If (2.2) has the trivial solution only then (2.5) has the trivial solution only which implies $\det M \neq 0$.

In the case of well posed boundary conditions the output y is uniquely determined by the input x. Hence a linear operator R such that $y = Rx$ is defined. The operator R is called *input output operator* of the system (2.2).

Theorem 2.1. *The matrix R of input output operator of the system of the form (2.1) with well posed boundary conditions is quasiseparable of order at most (m, m). Moreover let M be the matrix given by (2.3) and*

$$-M^{-1}M_1 = \begin{pmatrix} * & X_1 \\ * & X_2 \end{pmatrix}, \quad -M^{-1}M_2 = \begin{pmatrix} Y_1 & * \\ Y_2 & * \end{pmatrix}, \tag{2.6}$$

where matrices X_1, X_2, Y_1, Y_2 have the sizes $n_1 \times n_2$, $n_2 \times n_2$, $n_1 \times n_1$, $n_2 \times n_1$ respectively.

Then the elements

$$\begin{aligned} t_i &= \left[p_i(a_i^{\#} Y_1 a_{N,i-1}^{\times} + I_{n_1}) + g_i b_i^{\#} Y_2 a_{N,i-1}^{\times} \quad p_i a_i^{\#} X_1 + g_i b_i^{\#} X_2 \right], \; i = 2, \dots, N, \\ s_j &= \begin{bmatrix} q_j \\ b_{1j}^{\times} h_j \end{bmatrix}, \; j = 1, \dots, N-1, \\ l_k &= \begin{pmatrix} a_k & 0 \\ 0 & I_{n_2} \end{pmatrix}, \; k = 2, \dots, N-1; \end{aligned} \tag{2.7}$$

$$\begin{aligned} v_i &= \left[p_i a_i^{\#} Y_1 + g_i b_i^{\#} Y_2 \quad p_i a_i^{\#} X_1 b_{1,i+1}^{\times} + g_i(b_i^{\#} X_2 b_{1,i+1}^{\times} + I_{n_2}) \right], \; i = 1, \dots, N-1, \\ u_j &= \begin{bmatrix} a_{Nj}^{\times} q_j \\ h_j \end{bmatrix}, \; j = 2, \dots, N, \\ \delta_k &= \begin{pmatrix} I_{n_1} & 0 \\ 0 & b_k \end{pmatrix}, \; k = 2, \dots, N-1; \end{aligned} \tag{2.8}$$

$$\lambda_k = p_k a_k^{\#}(X_1 b_{1,k}^{\times} h_k + Y_1 a_{N,k}^{\times} q_k) + d_k + g_k b_k^{\#}(X_2 b_{1,k}^{\times} h_k + Y_2 a_{N,k}^{\times} q_k), \; k = 1, \dots, N \tag{2.9}$$

are generators of the matrix R.

Let us remark that q_N and h_1 are not determined from (2.1) and hence they are free. Since by the definition $a_{NN}^{\times} = 0$ and $b_{11}^{\times} = 0$ they may be chosen arbitrarily.

Proof. One can easily prove by induction that the solutions of the first and the second equations in (2.1) are given by

$$\chi_k = a_k^{\#}\chi_1 + f_k, \quad k = 1, \dots, N,$$

where $f_k = \sum_{j=1}^{k-1} a^{\times}_{kj} q_j x_j$ and

$$\eta_k = b^{\#}_k \eta_N + \phi_k, \quad k = N, \dots, 1,$$

where $\phi_k = \sum_{j=k+1}^{N} b^{\times}_{kj} h_j x_j$. By virtue of boundary conditions we obtain

$$M_1 \begin{pmatrix} \chi_1 \\ b^{\#}_1 \eta_N + \phi_1 \end{pmatrix} + M_2 \begin{pmatrix} a^{\#}_N \chi_1 + f_N \\ \eta_N \end{pmatrix} = 0$$

which implies

$$M \begin{pmatrix} \chi_1 \\ \eta_N \end{pmatrix} = -M_1 \begin{pmatrix} 0 \\ \phi_1 \end{pmatrix} - M_2 \begin{pmatrix} f_N \\ 0 \end{pmatrix}.$$

Hence it follows that

$$\chi_1 = X_1 \phi_1 + Y_1 f_N, \quad \eta_N = X_2 \phi_1 + Y_2 f_N.$$

Thus for the state space variables we have

$$\chi_k = a^{\#}_k (X_1 \phi_1 + Y_1 f_N) + f_k, \quad \eta_k = b^{\#}_k (X_2 \phi_1 + Y_2 f_N) + \phi_k, \quad k = 1, \dots, N.$$

Next for the output y we obtain

$$\begin{aligned} y_k &= p_k[a^{\#}_k (X_1 \phi_1 + Y_1 f_N) + f_k] + d_k x_k + g_k[b^{\#}_k (X_2 \phi_1 + Y_2 f_N) + \phi_k] = \\ &= p_k[a^{\#}_k (X_1 \sum_{j=1}^{N} b^{\times}_{1j} h_j x_j + Y_1 \sum_{j=1}^{N} a^{\times}_{Nj} q_j x_j) + \sum_{j=1}^{k-1} a^{\times}_{kj} q_j x_j] + d_k x_k + \\ &\quad + g_k[b^{\#}_k (X_2 \sum_{j=1}^{N} b^{\times}_{1j} h_j x_j + Y_2 \sum_{j=1}^{N} a^{\times}_{Nj} q_j x_j) + \sum_{j=k+1}^{N} b^{\times}_{kj} h_j x_j]. \end{aligned} \tag{2.10}$$

Hence follow representations for entries of the input output matrix R. In the case $N \geq i > j \geq 1$ using the relations $a^{\times}_{Nj} = a^{\times}_{N,i-1} a^{\times}_{i,j}$ we obtain

$$\begin{aligned} R_{ij} &= p_i[a^{\#}_i (X_1 b^{\times}_{1j} h_j + Y_1 a^{\times}_{Nj} q_j) + a^{\times}_{ij} q_j] + g_i b^{\#}_i (X_2 b^{\times}_{1j} h_j + Y_2 a^{\times}_{Nj} q_j) = \\ &= [p_i (a^{\#}_i Y_1 a^{\times}_{N,i-1} + I_{n_1}) + g_i b^{\#}_i Y_2 a^{\times}_{N,i-1}] a^{\times}_{ij} q_j + (p_i a^{\#}_i X_1 + g_i b^{\#}_i X_2) b^{\times}_{1j} h_j = t_i l^{\times}_{ij} s_j, \end{aligned}$$

where t_i, l_k, s_j are given by (2.7). Hence the matrix R is lower quasiseparable of order at most m with lower generators given by (2.7).

For $1 \leq i < j \leq N$ using the the relations $b^{\times}_{1j} = b^{\times}_{1,i+1} b^{\times}_{ij}$ we conclude that

$$\begin{aligned} R_{ij} &= p_i a^{\#}_i (X_1 b^{\times}_{1j} h_j + Y_1 a^{\times}_{Nj} q_j) + g_i[b^{\#}_i (X_2 b^{\times}_{1j} h_j + Y_2 a^{\times}_{Nj} q_j) + b^{\times}_{ij} h_j] = \\ &= (p_i a^{\#}_i Y_1 + g_i b^{\#}_i Y_2) a^{\times}_{Nj} q_j + [p_i a^{\#}_i X_1 b^{\times}_{1,i+1} + g_i (b^{\#}_i X_2 b^{\times}_{1,i+1} + I_{n_2})] b^{\times}_{ij} h_j = v_i \delta^{\times}_{ij} u_j, \end{aligned}$$

where v_i, δ_k, u_j are given by (2.8). Hence the matrix R is upper quasiseparable of order at most m with upper generators given by (2.8).

The desired relations (2.9) for diagonal entries λ_k of the matrix R follow from (2.10) directly.

Every quasiseparable of order (n_1, n_2) matrix R may be treated as an input output one for descriptor system of the form (2.1) of order $m = n_1 + n_2$.

Theorem 2.2. *Let R be a quasiseparable of order (n_1, n_2) matrix with generators p_i $(i = 2, \dots, N)$, q_j $(j = 1, \dots, N-1)$, a_k $(k = 2, \dots, N-1)$; g_i $(i = 1, \dots, N-1)$, h_j $(j = 2, \dots, N)$, b_k $(k = 2, \dots, N-1)$; d_k $(k = 1, \dots, N)$. Let a_1, b_N be arbitrary matrices and p_1, g_N be arbitrary vector rows of sizes $n_1 \times n_1$, $n_2 \times n_2$, n_1, n_2 correspondingly.*

Then R is input output matrix of the system

$$\begin{cases} \chi_{k+1} = a_k \chi_k + q_k x_k, \quad k = 1, \dots, N-1, \\ \eta_{k-1} = b_k \eta_k + h_k x_k, \quad k = N, \dots, 2, \\ y_k = p_k \chi_k + g_k \eta_k + d_k x_k, \quad k = 1, \dots N, \\ \chi_1 = 0, \quad \eta_N = 0. \end{cases} \tag{2.11}$$

Theorem 2.2 is an inversion of Theorem 2.1 without assumption on order of descriptor system.

Proof. The system (2.11) is a particular case of the system (2.1) with

$$M_1 = \begin{pmatrix} I_{n_1} & 0 \\ 0 & 0 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & 0 \\ 0 & I_{n_2} \end{pmatrix}.$$

It is easy to see that in this case all the matrices X_1, X_2, Y_1, Y_2 in (2.6) are zeroes and therefore in (2.7)-(2.9) we obtain $t_i = \begin{pmatrix} p_i & 0 \end{pmatrix}$, $t_i = \begin{pmatrix} 0 & g_i \end{pmatrix}$, $\lambda_i = d_i$. Hence by Theorem 2.1 it follows that the matrix with entries

$$R_{ij} = \begin{cases} p_i a_{ij}^{\times} q_j, & 1 \le j < i \le N, \\ d_i, & 1 \le i = j \le N, \\ g_i b_{ij}^{\times} h_j, & 1 \le i < j \le N \end{cases}$$

is an input output one for the system (2.11). But these elements are exactly the entries of the quasiseparable matrix R with generators p_i $(i = 2, \dots, N)$, q_j $(j = 1, \dots, N-1)$, a_k $(k = 2, \dots, N-1)$; g_i $(i = 1, \dots, N-1)$, h_j $(j = 2, \dots, N)$, b_k $(k = 2, \dots, N-1)$; d_k $(k = 1, \dots, N)$.

One can see that the coefficients of the system (2.11) are exactly the generators of its input output matrix.

3. Characteristic Properties

In this section we analyze in detail the properties of quasiseparable matrices. At first we show that quasiseparability is equivalent to some recursive relations for maximal submatrices of lower triangular and upper triangular parts.

Lemma 3.1. *Let R be a matrix of size $N \times N$ with lower generators p_i $(i = 2, \dots, N)$, q_j $(j = 1, \dots, N-1)$, a_k $(k = 2, \dots, N-1)$ of order n. Let us define matrices Q_k $(k =$

$1,\dots,N-1$) *of sizes* $n\times k$ *forward recursively and matrices* P_k ($k=N,\dots,2$) *of sizes* $(N-k)\times n$ *backward recursively as follows:*

$$Q_1=q_1,\quad Q_k=\begin{pmatrix} a_kQ_{k-1} & q_k\end{pmatrix},\quad k=2,\dots,N-1; \tag{3.1}$$

$$P_N=p_N,\quad P_k=\begin{pmatrix} p_k \\ P_{k+1}a_k\end{pmatrix},\quad k=N-1,\dots,2. \tag{3.2}$$

Then for maximal submatrices of the lower triangular part of the matrix R *the following representations are valid:*

$$R(k+1:N,1:k)=P_{k+1}Q_k,\quad k=1,\dots,N-1. \tag{3.3}$$

Proof. The successive application of (3.1) yields

$$\begin{aligned} Q_k=\begin{pmatrix} a_kQ_{k-1} & q_k\end{pmatrix}=\begin{pmatrix} a_ka_{k-1}Q_{k-2} & a_kq_{k-1} & q_k\end{pmatrix}=\dots \\ =\begin{pmatrix} a^{\times}_{k+1,1}q_1 & \dots & a^{\times}_{k+1,k-1}q_{k-1} & q_k\end{pmatrix}. \end{aligned} \tag{3.4}$$

Similarly using (3.2) we obtain

$$P_{k+1}=\begin{pmatrix} p_{k+1} \\ P_{k+2}a_{k+1}\end{pmatrix}=\begin{pmatrix} p_{k+1} \\ p_{k+2}a_{k+1} \\ P_{k+3}a_{k+2}a_{k+1}\end{pmatrix}=\begin{pmatrix} p_{k+1} \\ p_{k+2}a^{\times}_{k+2,k} \\ \vdots \\ p_Na^{\times}_{N,k}\end{pmatrix}. \tag{3.5}$$

Moreover the relation (1.1) yields

$$R(k+1:N,1:k)=\begin{pmatrix} p_{k+1}a^{\times}_{k+1,1}q_1 & \dots & p_{k+1}q_k \\ \vdots & \ddots & \vdots \\ p_Na^{\times}_{N,1}q_1 & \dots & p_Na^{\times}_{N,k}q_k\end{pmatrix},\quad k=1,\dots,N-1.$$

Then taking into consideration the equalities $a^{\times}_{m,t}=a^{\times}_{m,k}a^{\times}_{k+1,t}$ for $m>k>t$ one can conclude that

$$R(k+1:N,1:k)=\begin{pmatrix} p_{k+1} \\ p_{k+2}a^{\times}_{k+2,k} \\ \vdots \\ p_Na^{\times}_{N,k}\end{pmatrix}\cdot\begin{pmatrix} a^{\times}_{k+1,1}q_1 & \dots & a^{\times}_{k+1,k-1}q_{k-1} & q_k\end{pmatrix}=P_{k+1}Q_k.$$

Lemma 3.2. *Let p_i $(i = 2, \dots, N)$ be n-dimensional rows, q_j $(j = 1, \dots, N-1)$ n-dimensional columns, a_k $(k = 2, \dots, N-1)$ matrices of size $n \times n$. Let us define by the recursions (3.1), (3.2) the matrices Q_k $(k = 1, \dots N-1)$ of sizes $n \times k$ and the matrices P_k $(k = N, \dots, 2)$ of sizes $(N-k) \times n$. For a matrix R of size $N \times N$ let the relations (3.3) hold.*

Then p_i $(i = 2, \dots, N)$, q_j $(j = 1, \dots, N-1)$, a_k $(k = 2, \dots, N-1)$ are lower generators for the matrix R.

Proof. Let us consider an arbitrary element $R_{ij}, i > j$ of the lower triangular part of the matrix R. This element is the j-th entry in the first row of the submatrix $R(i : N, 1 : i-1)$. From (3.3) we conclude that $R(i : N, 1 : i-1) = P_i Q_{i-1}$. As was proved above the recursions (3.1), (3.2) for the matrices P_k, Q_k imply (3.4), (3.5). Thus we obtain

$$R(i : N, 1 : i-1) = \begin{pmatrix} p_i \\ p_{i+1} a^{\times}_{i+1,i-1} \\ \vdots \\ p_N a^{\times}_{N,i-1} \end{pmatrix} \cdot \begin{pmatrix} a^{\times}_{i,1} q_1 & \dots & a^{\times}_{i,i-2} q_{i-2} & q_{i-1} \end{pmatrix} =$$

$$= \begin{pmatrix} p_i a^{\times}_{i,1} q_1 & \dots & p_i q_{i-1} \\ \vdots & \ddots & \vdots \\ p_N a^{\times}_{N,1} q_1 & \dots & p_N a^{\times}_{N,i-1} q_{i-1} \end{pmatrix}.$$

In particular we have

$$R(i, 1 : i-1) = p_i \begin{pmatrix} a^{\times}_{i,1} q_1 & \dots & a^{\times}_{i,j} q_j & \dots & q_{i-1} \end{pmatrix}.$$

The j-th entry of this row is $p_i a^{\times}_{ij} q_j$ which means (1.1). Thus p_i, q_j, a_k are lower generators of R.

Similarly one can prove the following assertions concerning the upper triangular part of the matrix R.

Lemma 3.3. *Let R be a matrix of size $N \times N$ with upper generators g_i $(i = 1, \dots, N-1)$, h_j $(j = 2, \dots, N)$, b_k $(k = 2, \dots, N-1)$ of order n. Let us define matrices G_k $(k = 1, \dots, N-1)$ of sizes $k \times n$ forward recursively and matrices H_k $(k = N, \dots, 2)$ of sizes $n \times (N-k)$ backward recursively as follows:*

$$G_1 = g_1, \quad G_k = \begin{pmatrix} G_{k-1} b_k \\ g_k \end{pmatrix}, \quad k = 2, \dots, N-1; \tag{3.6}$$

$$H_N = h_N, \quad H_k = \begin{pmatrix} h_k & b_k H_{k+1} \end{pmatrix}, \quad k = N-1, \dots, 2. \tag{3.7}$$

Then for maximal submatrices of the upper triangular part of the matrix R the following representations are valid:

$$R(1 : k, k+1 : N) = G_k H_{k+1}, \quad k = 1, \dots, N-1. \tag{3.8}$$

Lemma 3.4. *Let g_i $(i = 1, \ldots, N-1)$ be n-dimensional rows, h_j $(j = 2, \ldots, N)$ n-dimensional columns, b_k $(k = 2, \ldots, N-1)$ matrices of size $n \times n$. Let us define by the recursions (3.6), (3.7) the matrices G_k $(k = 1, \ldots N-1)$ of sizes $k \times n$ and the matrices H_k $(k = N, \ldots, 2)$ of sizes $n \times (N-k)$. For a matrix R of size $N \times N$ let the relations (3.8) hold.*

Then g_i $(i = 1, \ldots, N-1)$, h_j $(j = 2, \ldots, N)$, b_k $(k = 2, \ldots, N-1)$ are upper generators for the matrix R.

Next we show using Lemmas 1-4 that quasiseparability of a matrix may be expressed in terms of rank of maximal submatrices of lower triangular and upper triangular parts.

Theorem 3.5. *A matrix R is lower quasiseparable of order n_1 if and only if every submatrix of R entirely located in the lower triangular part of R has rank n_1 at most and at least one of such submatrices has rank equal to n_1.*

A matrix R is upper quasiseparable of order n_2 if and only if every submatrix of R entirely located in the upper triangular part of R has rank n_2 at most and at least one of such submatrices has rank equal to n_2.

Proof. It is sufficient to prove the assertion of the theorem for the lower triangular part of the matrix R.

Assume that every submatrix of R entirely located in the lower triangular part of R has rank at most n_1. In particular for maximal submatrices we have

$$\operatorname{rank} R(k+1:N, 1:k) = r_k \leq n_1, \quad k = 1, \ldots, N-1. \tag{3.9}$$

Let us show that the matrix R has lower generators of order n_1.

The relation (3.9) yields for every matrix $R(k+1:N, 1:k)$ of the size $(N-k) \times k$ the representation

$$R(k+1:N, 1:k) = V_{k+1} W_k, \tag{3.10}$$

where V_{k+1} is a $(N-k) \times r_k$ matrix, W_k is a $r_k \times k$ matrix and $\operatorname{rank} V_{k+1} = \operatorname{rank} W_k = r_k$. One can add zero columns to V_{k+1} and zero rows to W_k in order to obtain $(N-k) \times n_1$ matrices $P_{k+1} = \begin{bmatrix} V_{k+1} & 0 \end{bmatrix}$ and $n_1 \times k$ matrices $Q_k = \begin{bmatrix} W_k \\ 0 \end{bmatrix}$. It easily follows from (3.10) that P_{k+1}, Q_k satisfy (3.3). Let p_k be the first row of P_k and q_k be the last column of Q_k. We should prove that there exist matrices a_k of size $n_1 \times n_1$ such that (3.1), (3.2) hold. Then by Lemma 3.2 it will follow that p_i $(i = 2, \ldots, N)$, q_j $(j = 1, \ldots, N-1)$, a_k $(k = 2, \ldots, N-1)$ are lower generators of R.

For the previous block $R(k:N, 1:k-1)$ we have

$$R(k:N, 1:k-1) = V_k W_{k-1},$$

where $\operatorname{rank} R(k:N, 1:k-1) = r_{k-1} \leq n_1$, V_k is a $(N-k+1) \times r_{k-1}$ matrix, W_{k-1} is a $r_{k-1} \times (k-1)$ matrix and $\operatorname{rank} V_k = \operatorname{rank} W_{k-1} = r_{k-1}$.

Let v_k be the first row of the matrix V_k and w_k be the last column of the matrix W_k. Then one can write down $V_k = \begin{pmatrix} v_k \\ V'_k \end{pmatrix}$, $W_k = (\, W'_k \quad w_k \,)$ and obtain

$$R(k:N,1:k-1) = \begin{pmatrix} v_k W_{k-1} \\ V'_k W_{k-1} \end{pmatrix}, \quad R(k+1:N,1:k) = (\, V_{k+1}W'_k \quad V_{k+1}w_k \,).$$

The submatrix $R(k+1:N,1:k-1)$ is a common part of the blocks $R(k:N,1:k-1)$ $R(k+1:N,1:k)$. For this part we have two representations and thus one can conclude that

$$V'_k W_{k-1} = V_{k+1} W'_k. \tag{3.11}$$

Let $\tilde{V}_{k+1}$ be such a $r_k \times (N-k)$ matrix that $\tilde{V}_{k+1}V_{k+1} = I_{r_k}$ and $\tilde{W}_{k-1}$ be such a $(k-1) \times r_{k-1}$ matrix that $W_{k-1}\tilde{W}_{k-1} = I_{r_{k-1}}$. Multiplying (3.11) by $\tilde{V}_{k+1}$ from the left and by $\tilde{W}_{k-1}$ from the right we obtain

$$(\tilde{V}_{k+1}V'_k)W_{k-1} = W'_k, \quad V'_k = V_{k+1}(W'_k\tilde{W}_{k-1}), \quad \tilde{V}_{k+1}V'_k = W'_k\tilde{W}_{k-1}.$$

Set $a'_k = \tilde{V}_{k+1}V'_k = W'_k\tilde{W}_{k-1}$. The matrix a'_k here has the sizes $r_k \times r_{k-1}$ and satisfies the relations

$$V'_k = V_{k+1}a'_k, \quad W'_k = a'_k W_{k-1}. \tag{3.12}$$

Next one can set

$$a_k = \begin{pmatrix} a'_k & 0_{r_k \times (n-r_k)} \\ 0_{(n-r_k)\times r_{k-1}} & 0_{(n-r_k)\times(n-r_{k-1})} \end{pmatrix}.$$

Next one can write down $P_k = \begin{pmatrix} p_k \\ P'_k \end{pmatrix}$, $Q_k = (\, Q'_k \quad q_k \,)$. From (3.12) we conclude that

$$P_{k+1}a_k = [\, V_{k+1} \quad 0 \,]\, a_k = [\, V_{k+1}a'_k \quad 0 \,] = [\, V'_k \quad 0 \,] = P'_k,$$
$$a_k Q_{k-1} = a_k \begin{bmatrix} W_{k-1} \\ 0 \end{bmatrix} = \begin{bmatrix} a'_k W_{k-1} \\ 0 \end{bmatrix} = \begin{bmatrix} W'_k \\ 0 \end{bmatrix} = Q'_k,$$

which implies (3.1), (3.2).

Assume that there exists a submatrix R^0 of R entirely located in the lower triangular part such that $\operatorname{rank} R^0 = n_1$. The matrix R^0 is a part of a certain $R(k_0+1:N,1:k_0)$ and using (3.9) we obtain

$$\operatorname{rank} R(k_0+1:N,1:k_0) = n_1. \tag{3.13}$$

One can conclude from here that n_1 is the minimal order of generators of the matrix R, that is R is lower quasiseparable of order n_1. Indeed if it is not a case we obtain by Lemma 3.1 that every submatrix $R(k+1:N,1:k)$ $(k=1,\ldots,N-1)$ may be represented in the form (3.3), where P_{k+1} and Q_k has the sizes $(N-k)\times n'$ and $n'\times k$ correspondingly and $n'<n_1$. Hence follows that $\operatorname{rank} R(k_0+1:N,1:k_0)<n_1$ which contradicts (3.13).

Let R be a lower quasiseparable of order n_1 matrix. Then any submatrix $R(k+1:N,1:k)$, $k=1,\dots,N-1$ by Lemma 1 has the form $R(k+1:N,1:k)=P_{k+1}Q_k$, where P_k and Q_k are matrices with the sizes $(N-k)\times n_1$ and $n_1\times k$ correspondingly. Hence it follows that $\operatorname{rank} R(k+1:N,1:k)\leq n_1$. Every submatrix $\tilde{R}$ of R entirely located in the lower triangular part of R is a submatrix of a certain $R(k_0+1:N,1:k_0)$. Therefore $\operatorname{rank}\tilde{R}\leq \operatorname{rank} R(k_0+1:N,1:k_0)\leq n_1$. Moreover at least one of the submatrices $R(k+1:N,1:k)=P_{k+1}Q_k$ has rank n_1. Indeed if it is not the case then for every $k=1,\dots,N-1$ we have $\operatorname{rank} R(k+1:N,1:k)\leq n'<n_1$ which as has been proved above implies that the matrix R is lower quasiseparable of order $\leq n'$ which is a contradiction.

4. Multiplication

We consider here the properties of the product of quasiseparable matrices and the product of a quasiseparable matrix by a vector. At first we show that the product of two lower (upper) quasiseparable matrices is lower (upper) quasiseparable of order the sum of the orders of the factors at most.

Theorem 4.1. *Let R_1, R_2 be matrices of sizes $N\times N$ which are lower quasiseparable of orders m_1, n_1 correspondingly. Then the product R_1R_2 is lower quasiseparable of order at most m_1+n_1.*

Let R_1, R_2 be matrices of sizes $N\times N$ which are upper quasiseparable of orders m_2, n_2 correspondingly. Then the product R_1R_2 is upper quasiseparable of order at most m_2+n_2.

Proof. It is sufficient to prove the assertion of the theorem for the case of lower quasiseparable matrices.

For any $k=1,\dots,N-1$ one can write down each of the matrices R_1, R_2, R_1R_2 in the form

$$R_1=\begin{pmatrix} A_k^1 & * \\ L_k^1 & B_{k+1}^1\end{pmatrix},\quad R_2=\begin{pmatrix} A_k^2 & * \\ L_k^2 & B_{k+1}^2\end{pmatrix},\quad R_1R_2=\begin{pmatrix} X_k & * \\ Z_k & *\end{pmatrix},$$

where A_k^1, A_k^2, X_k are principal leading matrices of size $k\times k$. From the condition of the theorem and Theorem 3.5 it follows that $\operatorname{rank} L_k^1\leq m_1$, $\operatorname{rank} L_k^2\leq n_1$. Moreover the equality

$$Z_k=L_k^1A_k^2+B_{k+1}^1L_k^2$$

holds and therefore

$$\operatorname{rank} Z_k\leq \operatorname{rank}(L_k^1A_k^2)+\operatorname{rank}(B_{k+1}^1L_k^2)\leq \operatorname{rank} L_k^1+\operatorname{rank} L_k^2\leq m_1+n_1.$$

Thus the assertion of the theorem follows by Theorem 3.5.

Next we show how generators of the product of two quasiseparable matrices may be expressed explicitly via generators of the factors.

Theorem 4.2. *Let R_1, R_2 be matrices of sizes $N \times N$ which are quasiseparable of orders (m_1, m_2) (n_1, n_2) respectively with generators p_i^1 $(i = 2, \dots, N)$, q_j^1 $(j = 1, \dots, N-1)$, a_k^1 $(k = 2, \dots, N-1)$; g_i^1 $(i = 1, \dots, N-1)$, h_j^1 $(j = 2, \dots, N)$, b_k^1 $(k = 2, \dots, N-1)$; d_k^1 $(k = 1, \dots, N)$ and p_i^2 $(i = 2, \dots, N)$, q_j^2 $(j = 1, \dots, N-1)$, a_k^2 $(k = 2, \dots, N-1)$; g_i^2 $(i = 1, \dots, N-1)$, h_j^2 $(j = 2, \dots, N)$, b_k^2 $(k = 2, \dots, N-1)$; d_k^2 $(k = 1, \dots, N)$ correspondingly. Then the generators t_i $(i = 2, \dots, N)$, s_j $(j = 1, \dots, N-1)$, l_k $(k = 2, \dots, N-1)$; v_i $(i = 1, \dots, N-1)$, u_j $(j = 2, \dots, N)$, δ_k $(k = 2, \dots, N-1)$; λ_k $(k = 1, \dots, N)$ of the matrix Q may be given as follows:*

$$t_i = [\, p_i^1 \quad d_i^1 p_i^2 + g_i^1 \psi_i a_i^2 \,], \quad s_j = \begin{bmatrix} a_j^1 \varphi_j h_j^2 + q_j^1 d_j^2 \\ q_j^2 \end{bmatrix}, \tag{4.1}$$

$$l_i = \begin{pmatrix} a_i^1 & q_i^1 p_i^2 \\ 0 & a_i^2 \end{pmatrix}, \tag{4.2}$$

$$v_i = [\, g_i^1 \quad p_i^1 \varphi_i b_i^2 + d_i^1 g_i^2 \,], \quad u_j = \begin{bmatrix} h_j^1 d_j^2 + b_j^1 \psi_j q_j^2 \\ h_j^2 \end{bmatrix}, \tag{4.3}$$

$$\delta_i = \begin{pmatrix} b_i^1 & h_i^1 g_i^2 \\ 0 & b_i^2 \end{pmatrix}, \tag{4.4}$$

$$\lambda_i = p_i^1 \varphi_i h_i^2 + d_i^1 d_i^2 + g_i^1 \psi_i q_i^2, \tag{4.5}$$

where

$$\varphi_i = \sum_{k=1}^{i-1} (a_{ik}^1)^\times q_k^1 g_k^2 (b_{ki}^2)^\times, \quad \psi_i = \sum_{k=i+1}^{N} (b_{ik}^1)^\times h_k^1 p_k^2 (a_{ki}^2)^\times. \tag{4.6}$$

Let us remark that g_N^1, a_N^2, a_1^1, h_1^2, p_1^1, b_1^2, b_N^1, q_N^2 are not determined from the definition of generators and hence they are free. Since by the definition $\varphi_1 = 0$ and $\psi_N = 0$ mentioned above parameters may be chosen arbitrarily. We assume them to be zeroes.

Proof. The entries of matrices R_1, R_2 have the form

$$R_{i,j}^1 = \begin{cases} p_i^1 (a_{ij}^1)^\times q_j^1, & 1 \le j < i \le N, \\ d_i^1, & i = j, \\ g_i^1 (b_{ij}^1)^\times h_j^1, & 1 \le i < j \le N \end{cases}$$

and

$$R_{i,j}^2 = \begin{cases} p_i^2 (a_{ij}^2)^\times q_j^2, & 1 \le j < i \le N, \\ d_i^2, & i = j, \\ g_i^2 (b_{ij}^2)^\times h_j^2, & 1 \le i < j \le N \end{cases}$$

respectively. For the entries Q_{ij} of the product $Q = R_1 R_2$ we obtain the following relations.

For $i > j$ we have

$$Q_{ij} = \sum_{k=1}^{N} R^1_{ik} R^2_{kj} = \sum_{k=1}^{j-1} p^1_i (a^1_{ik})^\times q^1_k g^2_k (b^2_{kj})^\times h^2_j + p^1_i (a^1_{ij})^\times q^1_j d^j_2 +$$
$$+ \sum_{k=j+1}^{i-1} p^1_i (a^1_{ik})^\times q^1_k p^2_k (a^2_{kj})^\times q^2_j + d^1_i p^2_i (a^2_{ij})^\times q^2_j + \sum_{k=i+1}^{N} g^1_i (b^1_{ik})^\times h^1_k p^2_k (a^2_{kj})^\times q^2_j.$$

By the definition of $\times$ operation for $k < j$ we have

$$(a^1_{ik})^\times = a^1_{i-1} \cdots a^1_{k+1} = a^1_{i-1} \cdots a^1_{j+1} a^1_j a^1_{j-1} \cdots a^1_{k+1} = (a^1_{ij})^\times a^1_j (a^1_{jk})^\times$$

and similarly for $k > i$

$$(a^2_{kj})^\times = a^2_{k-1} \cdots a^2_{j+1} = a^2_{k-1} \cdots a^2_{i+1} a^2_i a^2_{i-1} \cdots a^2_{j+1} = (a^2_{ki})^\times a^2_i (a^2_{ij})^\times.$$

Thus we obtain

$$Q_{ij} = p^1_i (a^1_{ij})^\times [a^1_j (\sum_{k=1}^{j-1} (a^1_{jk})^\times q^1_k g^2_k (b^2_{kj})^\times) h^2_j + q^1_j d^j_2] +$$
$$+ p^1_i [\sum_{k=j+1}^{i-1} (a^1_{ik})^\times q^1_k p^2_k (a^2_{kj})^\times] q^2_j + [d^1_i p^2_i + g^1_i (\sum_{k=i+1}^{N} (b^1_{ik})^\times h^1_k p^2_k (a^2_{ki})^\times) a^2_i] (a^2_{ki})^\times q^2_j =$$
$$= p^1_i (a^1_{ij})^\times (a^1_j \varphi_j h^2_j + q^1_j d^j_2) + p^1_i \Lambda_{ij} q^2_j + (d^1_i p^2_i + g^1_i \psi_i a^2_i)(a^2_{ki})^\times q^2_j.$$

In the last expression φ_j, ψ_j are given by (4.6) and

$$\Lambda_{ij} = \sum_{k=j+1}^{i-1} (a^1_{ik})^\times q^1_k p^2_k (a^2_{kj})^\times.$$

We have the relation

$$Q_{ij} = r_i \begin{pmatrix} (a^1_{ij})^\times & \Lambda_{ij} \\ 0 & (a^2_{ij})^\times \end{pmatrix} s_j,$$

where r_i, s_j are given by (4.1). To obtain desired representation for the lower triangular part of the matrix Q it remains to check that

$$\begin{pmatrix} (a^1_{ij})^\times & \Lambda_{ij} \\ 0 & (a^2_{ij})^\times \end{pmatrix} = l^\times_{ij}, \tag{4.7}$$

where the matrices l_k are defined in (4.2). The proof is by induction by i. The case $i = j+1$ is trivial. Assume that for $k = j+1, \ldots, i$ the assertion has been proved. For $k = i+1$ we have

$$l^\times_{i+1,j} = l_i l^\times_{ij} = \begin{pmatrix} a^1_i & q^1_i p^2_i \\ 0 & a^2_i \end{pmatrix} \begin{pmatrix} (a^1_{ij})^\times & \Lambda_{ij} \\ 0 & (a^2_{ij})^\times \end{pmatrix} = \begin{pmatrix} (a^1_{i+1,j})^\times & a^1_i \Lambda_{ij} + q^1_i p^2_i (a^2_{ij})^\times \\ 0 & (a^2_{i+1,j})^\times \end{pmatrix}.$$

For the right upper element we have

$$a_i^1\Lambda_{ij}+q_i^1p_i^2(a_{ij}^2)^\times=a_i^1\sum_{k=j+1}^{i-1}(a_{ik}^1)^\times q_k^1p_k^2(a_{kj}^2)^\times+(a_{i+1,i}^1)^\times q_i^1p_i^2(a_{ij}^2)^\times=$$
$$=\sum_{k=j+1}^{i-1}(a_{i+1,k}^1)^\times q_k^1p_k^2(a_{kj}^2)^\times+(a_{i+1,i}^1)^\times q_i^1p_i^2(a_{ij}^2)^\times=$$
$$=\sum_{k=j+1}^{i}(a_{i+1,k}^1)^\times q_k^1p_k^2(a_{kj}^2)^\times=\Lambda_{i+1,j},$$

which completes the proof of (4.7).

For $i=j$ we have relations

$$\lambda_i=Q_{ii}=\sum_{k=1}^N R_{ik}^1R_{ki}^2=p_i^1(\sum_{k=1}^{i-1}(a_{ik}^1)^\times q_k^1g_k^2(b_{ki}^2)^\times)h_i^2+d_i^1d_i^2+g_i^1(\sum_{k=i+1}^N(b_{ik}^1)^\times h_k^1p_k^2(a_{ki}^2)^\times)q_i^2$$

from which (4.5) follows.

For $i<j$ we have

$$Q_{ij}=\sum_{k=1}^{j-1}p_i^1(a_{ik}^1)^\times q_k^1g_k^2(b_{kj}^2)^\times h_j^2+d_i^1g_i^2(b_{ij}^2)^\times h_j^2+$$
$$+\sum_{k=j+1}^{i-1}g_i^1(b_{ik}^1)^\times h_k^1g_k^2(b_{kj}^2)^\times h_j^2+g_i^1(b_{ij}^1)^\times h_j^1d_2^j+\sum_{k=i+1}^{N}g_i^1(b_{ik}^1)^\times h_k^1p_k^2(a_{kj}^2)^\times q_j^2.$$

By the definition of $\times$ operation for $k<i$ we have

$$(b_{kj}^2)^\times=b_{k+1}^2\cdots b_{j-1}^2=b_{k+1}^2\cdots b_{i-1}^2b_i^2b_{i+1}^2\cdots b_{j-1}^2=(b_{ki}^2)^\times b_i^2(b_{ij}^2)^\times$$

and similarly for $k>j$

$$(b_{ik}^1)^\times=b_{i+1}^1\cdots b_{k-1}^1=b_{i+1}^1\cdots b_{j-1}^1b_j^1b_{j+1}^1\cdots b_{k-1}^1=(b_{ij}^1)^\times b_j^1(b_{jk}^1)^\times.$$

Thus we obtain

$$Q_{ij}=[p_i^1(\sum_{k=1}^{j-1}(a_{ik}^1)^\times q_k^1g_k^2(b_{ki}^2)^\times)b_i^2+d_i^1g_i^2](b_{ij}^2)^\times h_j^2+$$
$$+g_i^1[\sum_{k=i+1}^{j-1}(b_{ik}^1)^\times h_k^1g_k^2(b_{kj}^2)^\times]h_j^2+g_i^1(b_{ij}^1)^\times[h_j^1d_j^2+b_j^1(\sum_{k=j+1}^{N}(b_{jk}^1)^\times h_k^1p_k^2(a_{kj}^2)^\times)q_j^2].$$

Using expressions φ_i, ψ_j from (4.6) and denoting $\Gamma_{ij} = \sum_{k=i+1}^{j-1}(b_{ik}^1)^\times h_k^1 p_k^2 (b_{kj}^2)^\times$ one can conclude that

$$Q_{ij} = (p_1^i \varphi_i b_i^2 + d_i^1 g_i^2)(b_{ij}^2)^\times h_j^2 + g_1^i \Gamma_{ij} h_j^2 + g_i^1 (b_{ij}^1)^\times (h_j^1 d_j^2 + b_j^1 \psi_j q_j^2) = \\ = v_i \begin{pmatrix} (b_{ij}^1)^\times & \Gamma_{ij} \\ 0 & (b_{ij}^2)^\times \end{pmatrix} u_j,$$

where v_i, u_j are given by (4.3). One can check in the same way as in the proof of (4.7) that

$$\begin{pmatrix} (b_{ij}^1)^\times & \Gamma_{ij} \\ 0 & (b_{ij}^2)^\times \end{pmatrix} = \delta_{ij}^\times,$$

where δ_k are defined in (4.4). Thus the proof of the theorem is completed.

Based on Theorem 4.2 one can derive the following method for computing the generators of the product $Q = R_1 R_2$.

Algorithm 4.3.

1. Set $a_1^1 = 0$, $h_1^2 = 0$, $p_1^1 = 0$, $b_1^2 = 0$ (as was mentioned above these parameters could be chosen arbitrarily).

Set $\varphi_1 = 0_{m_1 \times n_2}$ and for $k = 1, \dots, N-1$ compute recursively

$$\alpha_k = a_k^1 \varphi_k h_k^2 + q_k^1 d_k^2, \quad \theta_k = p_k^1 \varphi_k b_k^2 + d_k^1 g_k^2, \\ \varphi_{k+1} = a_k^1 \varphi_k b_k^2 + q_k^1 g_k^2. \tag{4.8}$$

Set

$$s_k = \begin{bmatrix} \alpha_k \\ q_k^2 \end{bmatrix}, \quad v_k = [\, g_k^1 \quad \theta_k \,].$$

2. Set $g_N^1 = 0$, $a_N^2 = 0$, $b_N^1 = 0$, $q_N^2 = 0$ (as was mentioned above these parameters could be chosen arbitrarily).

Set $\psi_N = 0_{m_2 \times n_1}$ and for $k = N, \dots, 2$ compute recursively

$$\beta_k = d_k^1 p_k^2 + g_k^1 \psi_k a_k^2, \quad \eta_k = h_k^1 d_k^2 + b_k^1 \psi_k q_k^2, \\ \psi_{k-1} = b_k^1 \psi_k a_k^2 + h_k^1 p_k^2. \tag{4.9}$$

Set

$$t_k = [\, p_k^1 \quad \beta_k \,], \quad u_k = \begin{bmatrix} h_k^2 \\ \eta_k \end{bmatrix}.$$

3. For $k = 1, \dots, N$ compute

$$\lambda_k = p_k^1 \varphi_k h_k^2 + d_k^1 d_k^2 + g_k^1 \psi_k q_k^2.$$

4. For $k = 2, \dots, N-1$ compute $z_k = q_k^1 p_k^2$, $w_k = h_k^1 g_k^2$ and set

$$l_k = \begin{pmatrix} a_k^1 & z_k \\ 0 & a_k^2 \end{pmatrix}, \quad \delta_k = \begin{pmatrix} b_k^1 & w_k \\ 0 & b_k^2 \end{pmatrix}.$$

To justify this algorithm one should only check that auxiliary matrices φ_i, ψ_i satisfy recursive relations (4.8), (4.9). Indeed it follows directly from (4.6) that

$$\begin{aligned}\varphi_1 = 0, \quad \varphi_{i+1} &= \sum_{k=1}^{i} (a_{i+1,k}^1)^\times q_k^1 g_k^2 (b_{k,i+1}^2)^\times = \\ &= a_i^1 \Big(\sum_{k=1}^{i-1} (a_{ik}^1)^\times q_k^1 g_k^2 (b_{ki}^2)^\times\Big) b_i^2 + (a_{i+1,i}^1)^\times q_i^1 g_i^2 (b_{i,i+1}^2)^\times = \\ &\qquad = a_i^1 \varphi_i b_i^2 + q_i^1 g_i^2, \quad i = 1, 2, \dots N-1\end{aligned}$$

and

$$\begin{aligned}\psi_N = 0, \quad \psi_{i-1} &= \sum_{k=i}^{N} (b_{i-1,k}^1)^\times h_k^1 p_k^2 (a_{k,i-1}^2)^\times = \\ &= b_i^1 \Big(\sum_{k=i}^{N} (b_{ik}^1)^\times h_k^1 p_k^2 (a_{ki}^2)^\times\Big) a_i^2 + (b_{i,i+1}^1)^\times h_i^1 p_i^2 (a_{i+1,i}^2)^\times = \\ &\qquad = b_i^1 \psi_i a_i^2 + h_i^1 p_i^2, \quad i = N, N-1, \dots 2.\end{aligned}$$

Algorithm 4.3 does not contain embedding loops and therefore has linear complexity by N. The exact number of flops in this algorithm may be computed easily. Indeed consider for instance the computation of the element α_k. The operation $q_k^1 d_k^2$ is a product of a vector of size m_1 by a number and hence requires m_1 flops. The product $\varphi_k h_k^2$ as a product of a matrix of size $m_1 \times n_2$ by a vector of size n_2 requires $m_1 n_2$ flops. Next the product $a_k^1(\varphi_k h_k^2)$ will take m_1^2 flops. Thus the total complexity for computation of α_k is $m_1 + m_1 n_2 + m_1^2$. Similarly we obtain that computation of the variables θ_k, z_k, φ_{k+1}, β_k, η_k, w_k, ψ_{k-1}, λ_k requires correspondingly $n_2 + m_1 n_2 + n_2^2$, $m_1 n_1$, $m_1 n_2 + m_1 n_2^2 + m_1^2 n_2$, $n_1 + n_1 m_2 + n_1^2$, $m_2 + n_1 m_2 + m_2^2$, $m_2 n_2$, $n_1 m_2 + n_1 m_2^2 + n_1^2 m_2$, $m_1 n_2 + m_2 n_1 + n_1 + n_2 + 1$ flops. Thus the total complexity of Algorithm 4.3 is $(m_1^2 n_2 + m_1 n_2^2 + m_2^2 n_1 + n_1^2 m_2 + m_1^2 + n_1^2 + m_2^2 + n_2^2 + 3m_1 n_2 + 3m_2 n_1 + m_1 n_1 + m_2 n_2 + m_1 + n_1 + m_2 + n_2)(N-1) + (m_1 n_2 + m_2 n_1 + n_1 + n_2 + 1)N$ flops.

Let us consider now an algorithm for multiplication of a quasiseparable matrix by a vector and show that this algorithm has linear complexity by N in contrast to $O(N^2)$ in the case of a matrix of a general form. Let R be a quasiseparable matrix of order (n_1, n_2) with generators p_i $(i = 2, \dots, N)$, q_j $(j = 1, \dots, N-1)$, a_k $(k = 2, \dots, N-1)$; g_i $(i = 1, \dots, N-1)$, h_j $(j = 2, \dots, N)$, b_k $(k = 2, \dots, N-1)$; d_k $(k = 1, \dots, N)$. It means that entries of the matrix R have the form

$$R_{i,j} = \begin{cases} p_i a_{ij}^{\times} q_j, & 1 \le j < i \le N, \\ d_i, & i = j, \\ g_i b_{ij}^{\times} h_j, & 1 \le i < j \le N. \end{cases}$$

The product $y = Rx$ of the matrix R by the vector x is found as $y = y^L + y^D + y^U$, where $y^L = R_L x$, $y^D = R_D x$, $y^U = R_U x$ and R_L, R_D, R_U are correspondingly lower triangular, diagonal and upper triangular parts of the matrix R.

For y^L we have $y_1^L = 0$ and for $i \ge 2$

$$y_i^L = p_i z_i,$$

where

$$z_i = \sum_{j=1}^{i-1} a_{i,j}^{\times} q_j x_j.$$

Moreover z_i satisfies the recursive relations

$$z_{i+1} = \sum_{j=1}^{i} a_{i+1,j}^{\times} q_j x_j = a_i \sum_{j=1}^{i-1} a_{ij}^{\times} q_j x_j + a_{i+1,i} q_i x_i = a_i z_i + q_i x_i.$$

Similar relations hold for the upper triangular part, i.e. for the y^U.

Hence for $y = Rx$ we have the following algorithm.

Algorithm 4.4.

1. Set $a_1 = 0$. Start with $y_1^L = 0$, $z_1 = 0_{n_1 \times 1}$ and for $i = 2, \dots, N$ compute recursively

$$z_i = a_{i-1} z_{i-1} + q_{i-1} x_{i-1},$$
$$y_i^L = p_i z_i.$$

2. Compute for $i = 1, \dots, N$

$$y_i^D = d_i x_i.$$

3. Set $b_N = 0$. Start with $y_N^U = 0$, $w_N = 0_{n_2 \times 1}$ and for $i = N-1, \dots, 1$ compute recursively

$$w_i = b_{i+1} w_{i+1} + h_{i+1} x_{i+1},$$
$$y_i^U = g_i w_i.$$

4. Compute vector y

$$y = y^L + y^D + y^U.$$

Here we used that since $z_1 = 0$, $w_N = 0$ the parameters a_1, b_N may be chosen arbitrarily.

An easy calculation shows that this algorithm requires $(n_1^2 + 2n_1 + n_2^2 + 2n_2 + 1)(N-1) + 1$ flops.

5. Inversion

In this section we study inversion of quasiseparable matrices. As a basis we use relations between minors of a matrix and its inverse. From these relations we obtain easily that inverse to lower (upper) quasiseparable matrix is lower (upper) quasiseparable of the same order

Lemma 5.1. *Let R be invertible matrix of size N. Let for some integers k, m, n such that $1 \le m, k \le N-1$, $n \ge 0$, $m-k+n \ge 0$ the inequality*

$$\operatorname{rank} R(1:k, m+1:N) \le n \tag{5.1}$$

holds.

Then for the inverse matrix the inequality

$$\operatorname{rank} R^{-1}(1:m, k+1:N) \le m-k+n. \tag{5.2}$$

is valid.

Proof. Let r be an integer such that $r > m-k+n$ and R^0 be arbitrary square submatrix of the size $r \times r$ of the matrix $R^{-1}(1:m, k+1:N)$. The matrix R^0 may be represented as $R^0 = R^{-1}(\alpha, \beta)$, where α, β are sets of indices $\alpha = (i_1, \dots, i_r)$, $\beta = (j_1, \dots, j_r)$ such that $\alpha \subset (1:m)$, $\beta \subset (k+1:N)$. By the well known formula (see for instance [G, p. 17]) we have

$$|\det R^{-1}(\alpha, \beta)| = \frac{1}{|\det R|} |\det R(\beta', \alpha')|, \tag{5.3}$$

where α' and β' are the complements to α and β correspondingly in $(1, \dots, N)$. The matrix $R(\beta', \alpha')$ has the size $(N-r) \times (N-r)$. Moreover we have $\alpha' \supset \{1, \dots, k\}$, $\beta' \supset (m+1, \dots, N)$ from which follows that $R(\beta', \alpha')$ contains the matrix $R(1:k, m+1:N)$ with size $k \times (N-m)$ and rank at most n. Since the addition of the column or of the row to a matrix may increase its rank on one at most we conclude that

$$\operatorname{rank} R(\beta', \alpha') \le n + [(N-r)-k] + [(N-r)-(N-m)] = (N-r) - [r-(m+n-k)] < N-r.$$

Thus we obtain $\det R(\beta', \alpha') = 0$ and by virtue of (5.3) $\det R^0 = 0$ for any r such that $r > m-k+n$. Hence (5.2) follows.

Theorem 5.2. *Let R be a lower quasiseparable of order n_1 invertible matrix. Then the inverse matrix R^{-1} is lower quasiseparable of order n_1.*

Let R be an upper quasiseparable of order n_2 invertible matrix. Then the inverse matrix R^{-1} is upper quasiseparable of order n_2.

Proof. It is sufficient to consider the case of an upper quasiseparable matrix. By Theorem 3.5 if a matrix R of size N is upper quasiseparable of order n_2 then the relations

$$\operatorname{rank} R(1:k, k+1:N) \le n_2, \quad k = 1, \dots, N-1$$

hold which imply (5.1) with $n = n_2$, $k = 1, \ldots, N-1$, $m = k$. For the inverse matrix R^{-1} the application of Lemma 5.1 yields

$$\operatorname{rank} R^{-1}(1:k, k+1:N) \leq n_2, \quad k = 1, \ldots, N-1.$$

Hence by Lemma 5.1 the matrix R^{-1} is upper quasiseparable of order n_2', where $n_2' \leq n_2$. Applying the same arguments to the matrix R^{-1} we conclude that $n_2 \leq n_2'$ and thus R^{-1} is upper quasiseparable of order n_2.

The well known Asplund's theorem ([A]) concerning band matrices and inverses to them may be derived easily from Lemma 5.1 for the case of entries from $\mathbb{C}$. In accordance with Asplund's terminology a matrix $R = \{\rho_{ij}\}_{i,j=1}^{N}$ is called *upper band* of order n if its elements satisfy $\rho_{ij} = 0$ for $j > i + n$. A matrix $R = \{\rho_{ij}\}_{i,j=1}^{N}$ is called Green matrix of order n if every submatrix of R belonging to the part for which $j + n > i$ has rank n at most.

Theorem 5.3 (Asplund). *An invertible square matrix is an upper band matrix of order n if and only if its inverse is a Green matrix of order n.*

Proof. Let R be an upper band matrix of order n_2. It is equivalent to the assumption that R satisfies the relations

$$R(1:k, k+n_2+1:N) = 0, \quad k = 1, \ldots, N-n_2-1$$

which implies (5.1) with $n = 0$, $k = 1, \ldots, N-n_2-1$, $m = k + n_2$. In other words (5.1) holds for $m = n_2+1, \ldots, N-1$, $k = m - n_2$, $n = 0$. By virtue of Lemma 5.1 we conclude that

$$\operatorname{rank} R^{-1}(1:m, m-n_2+1:N) \leq n_2, \quad m = n_2+1, \ldots, N-1. \tag{5.4}$$

Let R^0 be a submatrix of the matrix R^{-1} belonging to the part for which $j+n > i$. R^0 is a submatrix of a certain $R^{-1}(1:m, m-n_2+1:N)$ and hence has rank at most n_2. Thus R^{-1} is a Green matrix of order n_2.

Let R^{-1} be a Green matrix of order n_2. It means that (5.4) holds. In other words the matrix R^{-1} satisfies (5.1) with $m = 1, \ldots, N-n_2-1$, $k = m + n_2$, $n = n_2$. Applying Lemma 5.1 to the matrix R^{-1} we obtain

$$\operatorname{rank} R(1:m, m+n_2+1:N) \leq m - (m+n_2) + n_2 = 0, \quad m = 1, \ldots, N-n_2-1$$

and thus the matrix R is an upper band of order n_2.

6. Inversion Formula and Algorithm in the Strongly Regular Case

We consider here the case when the matrix R is *strongly regular, that is all its principal leading minors are nonvanishing.* In this situation the generators of inverse matrix R^{-1} may be expressed explicitly via generators of the original matrix.

Theorem 6.1. *Let R be a strongly regular quasiseparable matrix of order (n_1, n_2) with generators p_i $(i = 2, \dots, N)$, q_j $(j = 1, \dots, N-1)$, a_k $(k = 2, \dots, N-1)$; g_i $(i = 1, \dots, N-1)$, h_j $(j = 2, \dots, N)$, b_k $(k = 2, \dots, N-1)$; d_k $(k = 1, \dots, N)$.*

Then generators t_i $(i = 2, \dots, N)$, s_j $(j = 1, \dots, N-1)$, l_k $(k = 2, \dots, N-1)$; v_i $(i = 1, \dots, N-1)$, u_j $(j = 2, \dots, N)$, δ_k $(k = 2, \dots, N-1)$; λ_k $(k = 1, \dots, N)$ of inverse matrix R^{-1} one can obtain as follows. The elements s_k, v_k, l_k, δ_k are given via forward algorithm

$$\gamma_1 = d_1, \quad s_1 = q_1 \gamma_1^{-1}, \quad v_1 = \gamma_1^{-1} g_1, \quad f_1 = s_1 g_1; \tag{6.1}$$

$$\gamma_k = d_k - p_k f_{k-1} h_k,$$

$$s_k = [q_k - a_k f_{k-1} h_k] \gamma_k^{-1}, \quad l_k = a_k - s_k p_k, \tag{6.2}$$

$$v_k = \gamma_k^{-1} [g_k - p_k f_{k-1} b_k], \quad \delta_k = b_k - h_k v_k, \tag{6.3}$$

$$f_k = a_k f_{k-1} b_k + [q_k - a_k f_{k-1} h_k] \cdot \gamma_k^{-1} \cdot [g_k - p_k f_{k-1} b_k], \quad k = 2, \dots N-1; \tag{6.4}$$

$$\gamma_N = d_N - p_N f_{N-1} h_N$$

and the elements λ_k, t_k, u_k are given via backward algorithm

$$\lambda_N = \gamma_N^{-1}, \quad t_N = -\lambda_N p_N, \quad u_N = -h_N \lambda_N, \quad z_N = -h_N t_N; \tag{6.5}$$

$$\lambda_k = \gamma_k^{-1} + v_k z_{k+1} s_k, \tag{6.6}$$

$$t_k = v_k z_{k+1} a_k - \lambda_k p_k, \quad u_k = b_k z_{k+1} s_k - h_k \lambda_k, \tag{6.7}$$

$$z_k = b_k z_{k+1} a_k - u_k p_k - h_k \lambda_k p_k - h_k t_k, \quad k = N-1, \dots 2; \tag{6.8}$$

$$\lambda_1 = \gamma_1^{-1} + v_1 z_2 s_1.$$

Here f_k, z_k are auxiliary matrices of sizes $n_1 \times n_2$ and $n_2 \times n_1$ respectively and γ_k is an auxiliary scalar variable.

Proof. For $k = 1, \dots, N-1$ let A_k be the principal leading submatrix of size $k \times k$ of the matrix R. Let us consider corresponding partitions of the matrix R

$$R = \begin{pmatrix} A_k & A_k'' \\ A_k' & B_{k+1} \end{pmatrix}.$$

From Lemma 3.1 we obtain $A_k' = P_{k+1} Q_k$, where P_k, Q_k are yielded recursively by the relations (3.1), (3.2). From Lemma 3.3 we have $A_k'' = G_k H_{k+1}$, where G_k, H_k are given by (3.6), (3.7). Thus we have representations

$$R = \begin{pmatrix} A_k & G_k H_{k+1} \\ P_{k+1} Q_k & B_{k+1} \end{pmatrix}. \tag{6.9}$$

The strong regularity of the matrix R implies that every A_k is invertible. Moreover from the well known inversion formula (see for instance [H, p. 466-467]) we obtain

$$R^{-1} = \begin{pmatrix} A_k^{-1} + (A_k^{-1} G_k)(H_{k+1} \tilde{B}_{k+1}^{-1} P_{k+1})(Q_k A_k^{-1}) & -(A_k^{-1} G_k)(H_{k+1} \tilde{B}_{k+1}^{-1}) \\ -(\tilde{B}_{k+1}^{-1} P_{k+1})(Q_k A_k^{-1}) & \tilde{B}_{k+1}^{-1} \end{pmatrix}, \tag{6.10}$$

where $\tilde{B}_{k+1} = B_{k+1} - P_{k+1}(Q_k A_k^{-1} G_k) H_{k+1}$.

Let us introduce the notations

$$V_k = A_k^{-1} G_k, \quad S_k = Q_k A_k^{-1}, \quad f_k = Q_k A_k^{-1} G_k;$$
$$U_k = -H_k \tilde{B}_k^{-1}, \quad T_k = -\tilde{B}_k^{-1} P_k, \quad z_k = H_k \tilde{B}_k^{-1} P_k.$$

Then (6.10) turns into

$$R^{-1} = \begin{pmatrix} A_k^{-1} + V_k z_{k+1} S_k & V_k U_{k+1} \\ T_{k+1} S_k & \tilde{B}_{k+1}^{-1} \end{pmatrix}. \tag{6.11}$$

Let us consider the matrices S_k, V_k, f_k of sizes $n_1 \times k$, $k \times n_2$, $n_1 \times n_2$ respectively. Let s_k, v_k be the last column and the last row of the matrices S_k, V_k correspondingly. For $k = 1$ we have $S_1 = s_1$, $V_1 = v_1$ and moreover $A_1 = d_1$, $Q_1 = q_1$, $G_1 = g_1$ from which (6.1) directly follows. For $k \geq 2$ we have the following. Changing k by $k-1$ in (6.9) we obtain

$$R = \begin{pmatrix} A_{k-1} & G_{k-1} H_k \\ P_k Q_{k-1} & B_k \end{pmatrix}.$$

Hence and from (6.9) it follows that

$$A_k = \begin{pmatrix} A_{k-1} & G_{k-1} H_k(1) \\ P_k(1) Q_{k-1} & B_k(1,1) \end{pmatrix},$$

where $P_k(1)$ is the first row of the matrix P_k, $H_k(1)$ is the first column of the matrix H_k, $B_k(1,1)$ is the upper left corner entry of the matrix B_k. It is obvious that $B_k(1,1) = d_k$. Moreover from (3.2) it follows $P_k(1) = p_k$ and (3.7) implies $H_k(1) = h_k$. Thus we obtain a representation similar to (6.9)

$$A_k = \begin{pmatrix} A_{k-1} & G_{k-1} h_k \\ p_k Q_{k-1} & d_k \end{pmatrix}.$$

Applying (6.11) to A_k we obtain

$$A_k^{-1} = \begin{pmatrix} A_{k-1}^{-1} + V_{k-1}(h_k \gamma_k^{-1} p_k) S_{k-1} & -V_{k-1} h_k \gamma_k^{-1} \\ -\gamma_k^{-1} p_k S_{k-1} & \gamma_k^{-1} \end{pmatrix}, \tag{6.12}$$

where $\gamma_k = d_k - p_k f_{k-1} h_k$.

Taking into consideration (3.1) and the equality $Q_{k-1} V_{k-1} = f_{k-1}$ we conclude that

$$\begin{aligned} S_k = Q_k A_k^{-1} &= \begin{pmatrix} a_k Q_{k-1} & q_k \end{pmatrix} A_k^{-1} = \\ &= \begin{pmatrix} a_k S_{k-1} + a_k f_{k-1}(h_k \gamma_k^{-1} p_k) S_{k-1} - q_k \gamma_k^{-1} p_k S_{k-1} & -a_k f_{k-1} h_k \gamma_k^{-1} + q_k \gamma_k^{-1} \end{pmatrix} = \\ &= \begin{pmatrix} \{a_k - [q_k - a_k f_{k-1} h_k] \gamma_k^{-1} p_k\} S_{k-1} & [q_k - a_k f_{k-1} h_k] \gamma_k^{-1} \end{pmatrix}. \end{aligned}$$

Similarly from (3.6) and the equality $S_{k-1}G_{k-1} = f_{k-1}$ we obtain

$$V_k = A_k^{-1}G_k = A_k^{-1}\begin{pmatrix} G_{k-1}b_k \\ g_k \end{pmatrix} =$$
$$= \begin{pmatrix} A_{k-1}^{-1}G_{k-1}b_k + V_{k-1}h_k\gamma_k^{-1}[p_kf_{k-1}b_k - g_k] \\ \gamma_k^{-1}[g_k - p_kf_{k-1}b_k] \end{pmatrix} =$$
$$= \begin{pmatrix} V_{k-1}\{b_k - h_k\gamma_k^{-1}[g_k - p_kf_{k-1}b_k]\} \\ \gamma_k^{-1}[g_k - p_kf_{k-1}b_k] \end{pmatrix}.$$

Finally for the matrix f_k we have

$$f_k = Q_kA_k^{-1}G_k = Q_kV_k = \begin{pmatrix} a_kQ_{k-1} & q_k \end{pmatrix}\begin{pmatrix} V_{k-1}\{b_k - h_k\gamma_k^{-1}[g_k - p_kf_{k-1}b_k]\} \\ \gamma_k^{-1}[g_k - p_kf_{k-1}b_k] \end{pmatrix} =$$
$$= a_kf_{k-1}b_k - a_kf_{k-1}h_k\gamma_k^{-1}[g_k - p_kf_{k-1}b_k] + q_k\gamma_k^{-1}[g_k - p_kf_{k-1}b_k] =$$
$$= a_kf_{k-1}b_k + [q_k - a_kf_{k-1}h_k]\gamma_k^{-1}[g_k - p_kf_{k-1}b_k].$$

Thus the elements $s_k,\ v_k,\ f_k$ satisfy relations (6.1)-(6.4). Moreover for the matrices $S_k,\ V_k$ we have recursions

$$S_1 = s_1, \quad S_k = \begin{pmatrix} l_kS_{k-1} & s_k \end{pmatrix}, \quad k = 2, \dots, N-1; \tag{6.13}$$

$$V_1 = v_1, \quad V_k = \begin{pmatrix} V_{k-1}\delta_k \\ v_k \end{pmatrix}, \quad k = 2, \dots, N-1, \tag{6.14}$$

where $l_k,\ \delta_k$ are given in (6.2), (6.3).

Let us consider the matrices $\tilde{B}_k^{-1},\ T_k,\ U_k$. Let λ_k be the left upper corner entry of the matrix $\tilde{B}_k^{-1}$. Notice that λ_k is the k-th entry of the main diagonal of the matrix R^{-1}. Let $t_k,\ u_k$ be the first row and the first column of the matrices $T_k,\ U_k$ correspondingly.

The formula (6.11) for $k = N-1$ yields $\tilde{B}_N^{-1} = \gamma_N^{-1} = \lambda_N$. Next from the definition of $U_k,\ T_k,\ z_k$ for $k = N$ we obtain (6.5).

By virtue of (6.11) we have

$$\tilde{B}_k^{-1} = \begin{pmatrix} A_k^{-1}(k,k) + v_kz_{k+1}s_k & v_ku_{k+1} \\ t_{k+1}s_k & \tilde{B}_{k+1}^{-1} \end{pmatrix} = \begin{pmatrix} \gamma_k^{-1} + v_kz_{k+1}s_k & v_ku_{k+1} \\ t_{k+1}s_k & \tilde{B}_{k+1}^{-1} \end{pmatrix}.$$

Hence follows the relations (6.6) for the diagonal entries λ_k and moreover using the equality $U_{k+1}P_{k+1} = -z_{k+1}$ we obtain

$$T_k = -\tilde{B}_k^{-1}P_k = -\begin{pmatrix} \lambda_k & v_kU_{k+1} \\ T_{k+1}s_k & \tilde{B}_{k+1}^{-1} \end{pmatrix}\begin{pmatrix} p_k \\ P_{k+1}a_k \end{pmatrix} = \begin{pmatrix} v_kz_{k+1}a_k - \lambda_kp_k \\ T_{k+1}(a_k - s_kp_k) \end{pmatrix}.$$

Further using $H_{k+1}T_{k+1} = -z_{k+1}$ we obtain

$$U_k = -H_k\tilde{B}_k^{-1} = -\begin{pmatrix} h_k & b_kH_{k+1} \end{pmatrix}\begin{pmatrix} \lambda_k & v_kU_{k+1} \\ T_{k+1}s_k & \tilde{B}_{k+1}^{-1} \end{pmatrix} =$$
$$= \begin{pmatrix} b_kz_{k+1}s_k - h_k\lambda_k & (b_k - h_kv_k)U_{k+1} \end{pmatrix}.$$

For the matrices z_k we have

$$z_k = H_k \tilde{B}_k^{-1} P_k = -H_k T_k = -\begin{pmatrix} h_k & b_k H_{k+1} \end{pmatrix} \begin{pmatrix} t_k \\ T_{k+1} l_k \end{pmatrix} = b_k z_{k+1} l_k - h_k t_k =$$
$$= b_k z_{k+1}(a_k - s_k p_k) - h_k t_k = b_k z_{k+1} a_k - [b_k z_{k+1} s_k - h_k \lambda_k] p_k -$$
$$- h_k \lambda_k p_k - h_k t_k = b_k z_{k+1} a_k - u_k p_k - h_k \lambda_k p_k - h_k t_k.$$

Thus the elements $t_k,\ u_k,\ z_k, \lambda_k$ satisfy relations (6.5)-(6.8). Moreover for the matrices $T_k,\ U_k$ we have recursions

$$T_N = t_N, \quad T_k = \begin{pmatrix} t_k \\ T_{k+1} l_k \end{pmatrix}, \quad k = N-1, \dots, 2; \tag{6.15}$$

$$U_N = u_N, \quad U_k = \begin{pmatrix} u_k & \delta_k U_{k+1} \end{pmatrix}, \quad k = N-1 \dots, 2. \tag{6.16}$$

From the relations (6.13), (6.15) by virtue of Lemma 3.2 it follows that the elements $t_i\ (i = 2, \dots, N),\ s_j\ (j = 1, \dots, N-1),\ l_k\ (k = 2, \dots, N-1)$ are lower generators of the inverse matrix R^{-1}. Similarly from the relations (6.14), (6.16) by virtue of Lemma 3.4 it follows that the elements $v_i\ (i = 2, \dots, N),\ u_j\ (j = 1, \dots, N-1),\ \delta_k\ (k = 2, \dots, N-1)$ are upper generators of the inverse matrix R^{-1}. The diagonal entries of R^{-1} are the elements λ_k which are given in (6.5), (6.6).Thus the elements $t_i\ (i = 2, \dots, N),\ s_j\ (j = 1, \dots, N-1),\ l_k\ (k = 2, \dots, N-1);\ v_i\ (i = 1, \dots, N-1),\ u_j\ (j = 2, \dots, N),\ \delta_k\ (k = 2, \dots, N-1);\ \lambda_k\ (k = 1, \dots, N)$ which are given by (6.1)-(6.8) are generators of the inverse matrix R^{-1}.

Note that in the case of diagonal plus semiseparable matrix the formulas for elements $s_k\ v_k$ in Theorem 6.1 coincide with expressions for a part of generators of the factors in LDU factorization of the matrix R in [GKK1]. Hence one can conclude that in this case some generators of the factors in LDU factorization and of the inverse matrix R^{-1} are the same. In the proof of Theorem 6.1 we clarify the meaning of the variable f_k which is used also in [GKK1]. The mentioned problems are related to results by Kailath and Sayed from [KS]. We intend to discuss them in detail in our next paper.

The computation of generators of the matrix R^{-1} may be performed as follows.

Algorithm 6.2.

1.1.Set $\gamma_1 = d_1$ and compute

$$\gamma_1' = \gamma_1^{-1}, \quad s_1 = q_1 \gamma_1', \quad v_1 = \gamma_1' g_1, \quad f_1 = s_1 g_1.$$

1.2.For $k = 2, \dots, N-1$ compute recursively

$$p_k' = p_k f_{k-1}, \quad h_k' = f_{k-1} h_k, \quad \gamma_k = d_k - p_k' h_k, \quad \gamma_k' = \gamma_k^{-1},$$
$$s_k' = q_k - a_k h_k', \quad s_k = s_k' \gamma_k',$$
$$v_k' = g_k - p_k' b_k, \quad v_k = \gamma_k' v_k',$$
$$l_k = a_k - s_k p_k, \quad \delta_k = b_k - h_k v_k,$$
$$f_k = a_k f_{k-1} b_k + s_k' v_k.$$

1.3.Compute

$$\gamma_N = d_N - p_N f_{N-1} h_N, \quad \gamma'_N = \gamma_N^{-1}.$$

Thus the elements $v_k,\ s_k,\ l_k,\ \delta_k,\ \gamma_k$ *are computed.*

2.1.Compute

$$\lambda_N = \gamma'_N, \quad t_N = -\lambda_N p_N, \quad u_N = -h_N \lambda_N, \quad z_N = -h_N t_N.$$

2.2.For $k = N-1, \dots, 2$ *compute recursively*

$$v''_k = v_k z_{k+1}, \quad s''_k = z_{k+1} s_k, \quad \lambda_k = \gamma'_k + v''_k s_k,$$
$$p''_k = \lambda_k p_k, \quad h''_k = h_k \lambda_k,$$
$$t_k = v''_k a_k - p''_k, \quad u_k = b_k s''_k - h''_k,$$
$$z_k = b_k z_{k+1} a_k - u_k p_k - h_k p''_k - h_k t_k.$$

2.3.Compute

$$\lambda_1 = \gamma'_1 + v_1 z_2 s_1.$$

Thus the elements $\lambda_k,\ t_k,\ u_k$ *are computed.*

An easy calculation shows that Algorithm 6.2 requires $(N-2)(2n_1^2 n_2 + 2n_1 n_2^2 + 10 n_1 n_2 + 2n_1^2 + 2n_2^2 + 3n_1 + 3n_2 + 1) + 4n_1 n_2 + 3n_1 + 3n_2 + 2$ flops.

Consequently using Algorithm 6.2 for generators of quasiseparable matrix R^{-1} and then applying Algorithm 4.4 to the product $x = R^{-1}y$ we obtain an algorithm for solution of linear equation $Rx = y$ of linear complexity.

7. The Case of Diagonal Plus Semiseparable Matrix

By the definition a matrix R is said to be *diagonal plus semiseparable of order* (n_1, n_2) if its entries are specified as follows:

$$R_{ij} = \begin{cases} p_i q_j, & 1 \le j < i \le N, \\ d_i, & 1 \le i = j \le N, \\ g_i h_j, & 1 \le i < j \le N. \end{cases} \tag{7.1}$$

Here p_i $(i = 2, \dots, N)$ and q_j $(j = 1, \dots, N-1)$ are correspondingly rows and columns of size n_1, g_i $(i = 1, \dots, N-1)$ and h_j $(j = 2, \dots, N)$ are rows and columns of size n_2. In other words the matrix R is composed of the lower triangular part of a matrix of rank n_1 at most and from the upper triangular part of another matrix of rank n_2 at most.

Let us remark that in general the inverse to diagonal plus semiseparable matrix is not diagonal plus semiseparable of the same order. Indeed the inverse to a band of order (n_1, n_2) matrix A with nonzero entries on external diagonals is diagonal plus semiseparable of order (n_1, n_2) matrix (see for instance [A]). But it is easy to see that for rather large sizes the matrix A cannot be diagonal plus semiseparable of order (n_1, n_2). However from the theorem from [GK] it follows that if for a matrix R with entries of the form (7.1) the numbers $l_k = d_k - p_k q_k$, $\delta_k = d_k - g_k h_k$, $k = 2, \dots, N-1$ are nonzeros then the inverse matrix R^{-1} is diagonal plus semiseparable of the same order as the matrix R.

Every diagonal plus semiseparable matrix is quasiseparable with generators p_i $(i = 2, \dots, N)$, q_j $(j = 1, \dots, N-1)$, $a_k = I_{n_1}$, $(k = 2, \dots, N-1)$; g_i $(i = 1, \dots, N-1)$, h_j $(j = 2, \dots, N)$, $b_k = I_{n_2}$ $(k = 2, \dots, N-1)$; d_k $(k = 1, \dots, N)$. Hence all the algorithms obtained above are applicable here.

Let R be a diagonal plus semiseparable strongly regular matrix. Then taking $a_k = I_{n_1}$, $b_k = I_{n_2}$ in Algorithm 6.2 we obtain the following method.

Algorithm 7.1.

Let R be a strongly regular matrix of the form (7.1). Then the generators t_i $(i = 2, \dots, N)$, s_j $(j = 1, \dots, N-1)$, l_k $(k = 2, \dots, N-1)$; v_i $(i = 1, \dots, N-1)$, u_j $(j = 2, \dots, N)$, δ_k $(k = 2, \dots, N-1)$; λ_k $(k = 1, \dots, N)$ of the quasiseparable matrix R^{-1} are given as follows.

1.1.Set $\gamma_1 = d_1$ and compute

$$\gamma_1' = \gamma_1^{-1}, \quad s_1 = q_1 \gamma_1', \quad v_1 = \gamma_1' g_1, \quad f_1 = s_1 g_1.$$

1.2.For $k = 2, \dots, N-1$ compute recursively

$$p_k' = p_k f_{k-1}, \quad h_k' = f_{k-1} h_k, \quad \gamma_k = d_k - p_k' h_k, \quad \gamma_k' = \gamma_k^{-1},$$
$$s_k' = q_k - h_k', \quad s_k = s_k' \gamma_k',$$
$$v_k' = g_k - p_k', \quad v_k = \gamma_k' v_k',$$
$$l_k = I_{n_1} - s_k p_k, \quad \delta_k = I_{n_2} - h_k v_k,$$
$$f_k = f_{k-1} + s_k' v_k.$$

1.3.Compute

$$\gamma_N = d_N - p_N f_{N-1} h_N, \quad \gamma_N' = \gamma_N^{-1}.$$

2.1.Compute

$$\lambda_N = \gamma_N', \quad r_N = -\lambda_N p_N, \quad u_N = -h_N \lambda_N, \quad z_N = -h_N t_N.$$

2.2.For $k = N-1, \dots, 2$ compute recursively

$$v_k'' = v_k z_{k+1}, \quad s_k'' = z_{k+1} s_k, \quad \lambda_k = \gamma_k' + v_k'' s_k,$$
$$p_k'' = \lambda_k p_k, \quad h_k'' = h_k \lambda_k,$$
$$t_k = v_k'' - p_k'', \quad u_k = s_k'' - h_k'',$$
$$z_k = z_{k+1} - u_k p_k - h_k p_k'' - h_k t_k.$$

2.3.Compute

$$\lambda_1 = \gamma_1' + v_1 z_2 s_1.$$

The complexity of this algorithm is $(N-2)(10n_1n_2+3n_1+3n_2+1)+4n_1n_2+3n_1+3n_2+2$ flops.

8. The Case of Band Matrix

By the definition a matrix $R = \{r_{ij}\}_{i,j=1}^{N}$ is said to be *band of order* (n_1, n_2) if $r_{ij} = 0$ for $i - j > n_1$ and $j - i > n_2$.

Every band of order (n_1, n_2) matrix is quasiseparable of order (n_1, n_2) at most. Its generators may be defined as follows. Let J_n be the square matrix of the size n of the form

$$\begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

and $e_n = [\,1 \quad 0 \quad \dots \quad 0\,]$ be the n-dimensional row. Let us set

$$p_i = e_{n_1},\ i = 2, \dots, N, \quad q_j = \begin{bmatrix} r_{j+1,j} \\ \vdots \\ r_{j+n_1,j} \end{bmatrix},\ j = 1, \dots, N-1,$$
$$a_k = J_{n_1},\ k = 2, \dots N-1;$$
$$g_i = [\,r_{i,i+1} \quad \dots \quad r_{i,i+n_2}\,],\ i = 1, \dots, N-1, \quad h_j = e_{n_2}^T,\ j = 2, \dots, N,$$
$$b_k = J_{n_2}^T,\ k = 2, \dots N-1;$$
$$d_k = r_{kk},\ k = 1, \dots N.$$

Here the entries r_{ij} for $i > N$ or $j > N$ are assumed to be zeros.

It is easy to check that such defined p_i $(i = 2, \dots, N)$, q_j $(j = 1, \dots, N-1)$, a_k $(k = 2, \dots, N-1)$; g_i $(i = 1, \dots, N-1)$, h_j $(j = 2, \dots, N)$, b_k $(k = 2, \dots, N-1)$; d_k $(k = 1, \dots, N)$ are generators of the matrix R. Indeed for $i > j$ we have

$$a_{ij}^{\times} = a_{i-1} \cdots a_{j+1} = J_{n_1}^{i-j-1}.$$

Hence for $0 < i - j \le n_1$ we obtain

$$p_i a_{ij}^{\times} q_j = p_i \begin{bmatrix} r_{ij} \\ * \end{bmatrix} = r_{ij}.$$

For $i - j > n_1$ we conclude that $p_i a_{ij}^{\times} q_j = p_i \cdot 0 \cdot q_j = 0$. For $j > i$ one can proceed similarly.

Let R be a band strongly regular matrix. In this case the following algorithm is obtained from Algorithm 6.2.

Algorithm 8.1.

Let $R = \{r_{ij}\}_{i,j=1}^N$ be a strongly regular band of order (n_1, n_2) matrix. Then generators t_i $(i = 2, \dots, N)$, s_j $(j = 1, \dots, N-1)$, l_k $(k = 2, \dots, N-1)$; v_i $(i = 1, \dots, N-1)$, u_j $(j = 2, \dots, N)$, δ_k $(k = 2, \dots, N-1)$; λ_k $(k = 1, \dots, N)$ of the quasiseparable matrix R^{-1} are given as follows.

1.1. Set $\gamma_1 = r_{11}$, $q_1 = \begin{bmatrix} r_{21} \\ \vdots \\ r_{n_1+1,1} \end{bmatrix}$, $g_1 = \begin{bmatrix} r_{12} & \dots & r_{1,n_2+1} \end{bmatrix}$ *and compute*

$$\gamma_1' = \gamma_1^{-1}, \quad s_1 = q_1 \gamma_1', \quad v_1 = \gamma_1' g_1, \quad f_1 = s_1 g_1.$$

1.2. For $k = 2, \dots, N-1$ perform the following operations:

1.2.1. Set

$$q_k = \begin{bmatrix} r_{k+1,k} \\ \vdots \\ r_{k+n_1,k} \end{bmatrix}, \quad g_k = \begin{bmatrix} r_{k,k+1} & \dots & r_{k,k+n_2} \end{bmatrix},$$

$$\tilde{h}_k = \begin{bmatrix} f_{k-1}(2,1) \\ \vdots \\ f_{k-1}(n_1,1) \\ 0 \end{bmatrix}, \ \tilde{p}_k = \begin{bmatrix} f_{k-1}(1,2) & \dots & f_{k-1}(1,n_2) & 0 \end{bmatrix}$$

and compute

$$s_k' = q_k - \tilde{h}_k, \quad v_k' = g_k - \tilde{p}_k, \tag{8.1}$$

$$\gamma_k = r_{k,k} - f_{k-1}(1,1), \tag{8.2}$$

$$\gamma_k' = \gamma_k^{-1}, \quad s_k = s_k' \gamma_k', \quad v_k = \gamma_k' v_k'.$$

1.2.2. Set

$$\tilde{f}_k = \begin{pmatrix} f_{k-1}(2,2) & \dots & f_{k-1}(2,n_2) & 0 \\ \vdots & \ddots & \vdots & \vdots \\ f_{k-1}(n_1,2) & \dots & f_{k-1}(n_1,n_2) & 0 \\ 0 & \dots & 0 & 0 \end{pmatrix}$$

and compute

$$f_k = \tilde{f}_k + s_k' v_k. \tag{8.3}$$

1.2.3. Set

$$l_k = \begin{pmatrix} -s_k(1) & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -s_k(n_1-1) & 0 & \dots & 1 \\ -s_k(n_1) & 0 & \dots & 0 \end{pmatrix}, \ \delta_k = \begin{pmatrix} -v_k(1) & \dots & -v_k(n_2-1) & -v_k(n_2) \\ 1 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 1 & 0 \end{pmatrix}. \tag{8.4}$$

1.3. Compute

$$\gamma_N = r_{N,N} - f_{N-1}(1,1), \quad \gamma'_N = \gamma_N^{-1}.$$

2.1. Set $\lambda_N = \gamma'_N$, $t_N = -\lambda_N e_{n_1}$, $u_N = -\lambda_N e_{n_2}^T$, $z_N = \lambda_N e_{n_2}^T e_{n_1}$.

2.2. For $k = N-1, \dots, 2$ *perform the following operations:*

2.2.1. Compute

$$v''_k = v_k z_{k+1}, \quad s''_k = z_{k+1} s_k, \quad \lambda_k = \gamma'_k + v''_k s_k.$$

2.2.2. Set

$$t_k = [\, -\lambda_k \quad v''_k(1) \quad \dots \quad v''_k(n_1-1) \,], \quad u_k = \begin{bmatrix} -\lambda_k \\ s''_k(1) \\ \vdots \\ s''_k(n_2-1) \end{bmatrix}. \tag{8.5}$$

2.2.3. Set

$$z_k = \begin{pmatrix} \lambda_k & -t_k(2) & \dots & -t_k(n_1) \\ -u_k(2) & z_{k+1}(1,1) & \dots & z_{k+1}(1, n_1-1) \\ \vdots & \vdots & \ddots & \vdots \\ -u_k(n_2) & z_{k+1}(n_2-1,1) & \dots & z_{k+1}(n_2-1, n_1-1) \end{pmatrix}. \tag{8.6}$$

2.3. Compute

$$\lambda_1 = \gamma'_1 + v_1 z_2 s_1.$$

To justify this algorithm notice that data of Algorithm 6.2 in the case under consideration may be expressed as follows. For the variables p'_k, h'_k we have

$$p'_k = [\, f_{k-1}(1,1) \quad \dots \quad f_{k-1}(1,n_2) \,], \quad h'_k = \begin{bmatrix} f_{k-1}(1,1) \\ \vdots \\ f_{k-1}(n_1,1) \end{bmatrix}.$$

Next one can introduce the variables $\tilde{p}_k = p'_k b_k$, $\tilde{h}_k = a_k h'_k$, $\tilde{f}_k = a_k f_{k-1} b_k$ and obtain relations (8.1) and (8.3). From the relations $d_k = r_{k,k}$, $p'_k h_k = p'_k(1) = f_{k-1}(1,1)$ the relation (8.2) follows. The relations (8.4) are obtained directly from $p_k = e_{n_1}$, $h_k = e_{n_2}^T$, $a_k = J_{n_1}$, $b_k = J_{n_2}$. Next for p''_k, h''_k we obtain $p''_k = \lambda_k e_{n_1}$, $h''_k = \lambda_k e_{n_2}^T$. Set $\tilde{v}_k = v''_k a_k$, $\tilde{s}_k = b_k s''_k$, $\tilde{z}_k = a_k z_{k+1} b_k$. We have

$$\tilde{v}_k = [\, 0 \quad v''_k(1) \quad \dots \quad v''_k(n_1-1) \,], \quad \tilde{s}_k = \begin{bmatrix} 0 \\ s''_k(1) \\ \vdots \\ s''_k(n_2-1) \end{bmatrix},$$

$$\tilde{z}_k = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & z_{k+1}(1,1) & \dots & z_{k+1}(1, n_1-1) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & z_{k+1}(n_2-1,1) & \dots & z_{k+1}(n_2-1, n_1-1) \end{pmatrix}.$$

For $t_k = \tilde{v}_k - p_k''$, $u_k = \tilde{s}_k - h_k''$ the relations (8.5) are obtained. Finally from

$$z_k = \tilde{z}_k - u_k p_k - h_k \lambda_k p_k - h_k t_k = \tilde{z}_k - u_k e_{n_1} - \lambda_k e_{n_1} e_{n_2}^T - e_{n_2}^T t_k$$

the relation (8.6) follows.

The complexity of Algorithm 8.1 is $(N-2)(3n_1 n_2 + 2n_1 + n_2 + 1) + 2n_1 n_2 + 2n_1 + n_2 + 2$ flops.

9. Numerical Experiments

As an illustration we present here the results of computer experiments with designed algorithms. We investigate their behavior in floating point arithmetic and compare them with other available algorithms. We solved linear systems $Rx = y$ for random values of input data p, q, g, h, d, y, a, b. The following algorithms were used:

(1)	GECP	Gaussian eliminations with complete pivoting.
(2)	GEPP	Gaussian eliminations with partial pivoting.
(3)	GE1	Algorithm 6.2, 4.4.
(4)	GKK	Gohberg-Kailath-Koltracht algorithm from [GKK1]
(5)	GK	algorithm derived in [EG2] using Gohberg-Kaashoek formula
(6)	GE	algorithm derived by the authors in [EG2] for diagonal plus semiseparable matrix of general form
(7)	GES	Algorithm 7.1, 4.4

All the algorithms (1)-(7) were implemented in the system MATLAB, version 4.2 with unit round-off error 2.2204×10^{-16}. The accuracy of the solutions obtained was estimated by the relations

$$\varepsilon = \frac{||x - x_{GECP}||}{||x_{GECP}||}, \quad \varepsilon_y = \frac{||Rx - y||}{||y||},$$

where x is the solution obtained by the corresponding algorithm, x_{GECP} is the solution obtained by the GECP method which we assume to be exact. The values of the input data we obtained by using the random-function. In each case the condition number $\kappa_2(R)$ of the original matrix was also computed.

In all experiments performed the input data were taken randomly. The values of elements of p, q, g, h, y, were chosen in the range of 0 to 10, the values of a, b were in the range of 0 to 1 and the values of the diagonal d were taken from the range of 0 to 100.

The data on time required by the above algorithms are also presented here. The authors have to make a proviso that the test programs were not completely optimized for time performance. At the same time these data can provide an approximation for the real complexities of the compared algorithms.

1. The first series of experiments was performed in the general situation. We compare here GEPP and GE1 algorithms. The results of computations are presented in Table 1.

Table 1. $n_1 = 2$, $n_2 = 2$

N	$\kappa_2(R)$	GEPP ε	GEPP ε_y	GE1 ε	GE1 ε_y
20	4e+3	1e-14	2e-14	4e-15	3e-14
50	2e+3	2e-14	6e-15	1e-14	3e-14
100	8e+4	1e-14	3e-14	1e-14	2e-13
150	5e+5	1e-15	1e-13	1e-13	3e-13
200	1e+6	1e-14	4e-13	1e-12	8e-12

The data on time required by these algorithms are presented in the following table.

Table 2. Time (seconds)

N	GEPP	GE1
20	1.32	0.46
50	11.16	0.92
100	97.10	1.92
150	270.92	2.86
200	1812.9	8.93

Thus one can conclude that for approximately the same accuracy the time required for the algorithm developed is essentially less than for the standard procedure.

2. In the second series we investigated the behavior of algorithms developed for the case of diagonal plus semiseparable matrix. The results are presented in Table 3.

Table 3.

N	$\kappa_2(R)$	GEPP ε	GEPP ε_y	GKK ε	GKK ε_y	GK ε	GK ε_y	GE ε	GE ε_y	GES ε	GES ε_y
20	1e+3	8e-15	9e-15	7e-15	1e-14	1e-14	6e-14	5e-15	9e-14	1e-14	6e-14
50	4e+3	3e-14	8e-15	1e-14	6e-14	5e-15	4e-14	6e-15	9e-14	5e-14	3e-13
100	6e+4	4e-14	5e-14	2e-14	6e-14	7e-10	7e-10	3e-14	2e-12	3e-13	9e-12
150	9e+4	1e-13	1e-13	5e-14	2e-13	5e-14	4e-13	6e-14	1e-11	1e-13	1e-11
200	7e+4	5e-14	7e-14	3e-14	2e-13	2e-14	3e-13	3e-14	6e-13	1e-14	3e-13

The corresponding data of time required are the following.

Table 4. Time (seconds)

N	GECP	GKK	GK	GE	GES
20	0.88	0.42	0.50	1.57	0.34
50	28.81	0.32	0.56	5.20	0.78
100	190.00	0.86	2.42	18.00	9.16
150	867.45	1.02	2.79	14.94	3.09
200	1471	1.35	3.31	22.18	4.98

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