Note Title 2025-02-28

- Vectorization
- Kronecker products
- Sylvester equations

$$\underset{m\times m}{A}\,\underset{m\times n}{X} - \underset{m\times n}{X}\,\underset{n\times n}{B} = \underset{m\times n}{C}$$

$$K \in \mathbb{C}^{mn\times mn} \qquad K = I\otimes A - B^T \otimes I$$

Bartels-Stewart algorithm:

1) $A = Q_A U_A Q_A^*$ $\qquad B^T = Q_B U_B Q_B^*$

   U_A, U_B upper triag, Q_A, Q_B unitary

2) Change of variables: $U_A Y - Y U_B^T = E$

3) back-substitution

- Complexity: $O(m^3+n^3)$ vs. $O(m^3n^3)$ for $lu(K)$
- Same idea works for $U_A Y - Y U_B = E$
- Same idea works for $X - AXB = C$ (Stein's equation)
  - Th: Stein's equation is uniquely solvable if and only if there are no $\lambda\in\Lambda(A)$, $\mu\in\Lambda(B)$ s.t. $\lambda\mu = 1$

- $O(m^3+n^3)$ solution algorithm
- Also for $AXB+CXD=E$, but not for
$$AXB+CXD+EXF=G$$

Conditioning of Sylvester equations:

$X \in \mathbb{C}^{m\times n}$
$\mathrm{vec}(X) \in \mathbb{C}^{mn}$

$$AX-XB=C \iff K\,\mathrm{vec}(X) = \mathrm{vec}(C)$$

$\|X\|_F = \|\mathrm{vec}(X)\|_2$, clearly (same elements in different order)

If one perturbs A,B,C to $\tilde{A},\tilde{B},\tilde{C}$ the solutions of $AX-XB=C$ and $\tilde{A}\tilde{X}-\tilde{X}\tilde{B}=\tilde{C}$ are related by

$$\frac{\|\mathrm{vec}(\tilde{X})-\mathrm{vec}(X)\|_2}{\|\mathrm{vec}(X)\|_2} \leq \mathrm{cond}(K)\cdot\left(\frac{\|\mathrm{vec}(\tilde{C})-\mathrm{vec}(C)\|_2}{\|\mathrm{vec}(C)\|_2}+\frac{\|\tilde{K}-K\|_2}{\|K\|_2}\right)$$

$$+O\left(\|\tilde{K}-K\|^2\right)$$

$$\Rightarrow \frac{\|\tilde{X}-X\|_F}{\|X\|_F} \leq \boxed{\mathrm{cond}(K)}\left(\frac{\|\tilde{C}-C\|_F}{\|C\|_F}+\frac{\|\tilde{A}-A\|+\|\tilde{B}-B\|}{\max(\|A\|,\|B\|)}\right)$$

$$\tilde{K} = I\otimes\tilde{A}-\tilde{B}^T\otimes I$$

$$\|\tilde{K}-K\| = \left\|I\otimes(\tilde{A}-A)-(\tilde{B}-B)^T\otimes I\right\| \leq \|I\otimes(\tilde{A}-A)\|+\|(\tilde{B}-B)^T\otimes I\|$$
$$=\|\tilde{A}-A\|+\|\tilde{B}-B\|$$

$\mathrm{cond}(K) = \|K\|\cdot\|K^{-1}\|$ plays a role

$$\|K\| = \|I \otimes A - B^T \otimes I\| \leq \|I \otimes A\| + \|B^T \otimes I\| \leq \|A\| + \|B\|$$

$\|K^{-1}\|$ is more complicated

$$\frac{1}{\|K^{-1}\|} = \sigma_{\min}(K) = \min_{v \neq 0} \frac{\|Kv\|}{\|v\|} = \boxed{\min_{\substack{Z \in \mathbb{C}^{m \times n} \\ Z \neq 0}} \frac{\|AZ - ZB\|_F}{\|Z\|_F}}$$

does not have an easy estimate

We call it $$\operatorname{sep}(A,B) := \boxed{\min_{\substack{Z \in \mathbb{C}^{m \times n} \\ Z \neq 0}} \frac{\|AZ - ZB\|_F}{\|Z\|_F}}$$

$\operatorname{Sep}(A,B)$ has a simpler expression if A, B are <u>normal</u>. A normal $\Leftrightarrow U_A$ is <u>diagonal</u> in $A = Q_A U_A Q_A^*$

B normal $\Leftrightarrow B^T$ normal $\Leftrightarrow U_B$ diagonal in $B^T = Q_B U_B Q_B^*$

Recall that a Schur decomposition of K is

$$K = I \otimes A - B^T \otimes I = (Q_B \otimes Q_A) \underbrace{(I \otimes U_A - U_B \otimes I)}_{\text{diagonal if } A,B \text{ normal}} (Q_B \otimes Q_A)^*$$

If U_A, U_B are diagonal, this is also an eigenvalue decomposition and an SVD (up to absolute values)

In particular, the singular values of K are

$$\{ |\lambda_i - \mu_j| \quad i = 1, \ldots m, \ j = 1, \ldots n \}$$

and $$\sigma_{\min}(K) = \min_{\substack{i=1,\ldots m \\ j=1,\ldots n}} |\lambda_i - \mu_j| .$$

$$\rightarrow \operatorname{sep}(A,B) = \sigma_{\min}(K) = \min_{\substack{i=1,\ldots m \\ j=1,\ldots n}} |\lambda_i - \mu_j| \quad \text{for } \underline{\text{normal}} \ A,B$$

For non-normal A, B, this is no longer true, but an inequality holds:

$$\sigma_{min}(K) \leq \min_{\lambda \in \Lambda(K)} |\lambda| = \min_{\substack{i=1,\dots m \\ j=1,\dots n}} |\lambda_i - \mu_j| .$$

Example: decoupling eigenvalues

Given $\begin{bmatrix} \lambda & c \\ 0 & \mu \end{bmatrix} \in \mathbb{C}^{2\times 2}$, if $\lambda \neq \mu$, is similar to

$\begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}$.

We can construct a similarity matrix explicitly:

$$\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \lambda & c \\ 0 & \mu \end{bmatrix}\begin{bmatrix} 1 & -x \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \lambda & c+x\mu \\ 0 & \mu \end{bmatrix}\begin{bmatrix} 1 & -x \\ 0 & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} \lambda & c+x\mu-\lambda x \\ 0 & \mu \end{bmatrix}$$

You can always choose x s.t. $c + x\mu - \lambda x = 0$, if $\mu \neq \lambda$

With Sylvester equations, we can do something similar with blocks:

$$\begin{array}{c|cc} & m & n \\ \hline m & A & C \\ n & 0 & B \end{array} \qquad A \in \mathbb{C}^{m\times m}, \; B \in \mathbb{C}^{n\times n}, \; C \in \mathbb{C}^{m\times n}$$

We would like to build a similarity transformation that transforms this matrix into $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$

$$\begin{bmatrix} I & -X \\ 0 & I \end{bmatrix}\begin{bmatrix} A & C \\ 0 & B \end{bmatrix}\begin{bmatrix} I & X \\ 0 & I \end{bmatrix} = \begin{bmatrix} A & C-XB \\ 0 & B \end{bmatrix}\begin{bmatrix} I & X \\ 0 & I \end{bmatrix} = \begin{bmatrix} A & AX-XB+C \\ 0 & B \end{bmatrix}$$

one the inverse of the other

If X satisfies $AX - XB + C = 0$ (Sylvester equation), then on the right we have
$$\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$$

We already know that the Sylv. equation is solvable if $\Lambda(A) \cap \Lambda(B) = \emptyset$.

Backward stability of the Bartels-Stewart algorithm:

We say that an algorithm is backward stable if it delivers a solution to a slightly perturbed problem:

$\tilde{X}$ computed in machine arithmetic satisfies
$\boxed{\hat{A}\tilde{X} - \tilde{X}\hat{B} = \hat{C}}$ for matrices $\hat{A}, \hat{B}, \hat{C}$ s.t.
$$\frac{\|\hat{A} - A\|}{\|A\|}, \frac{\|\hat{B} - B\|}{\|B\|}, \frac{\|\hat{C} - C\|}{\|C\|} = O(mnu)$$
$u \approx 10^{-16}$ machine precision

Many algorithms used in lin. algebra are backward stable:

- The QR/Francis algorithm computes $\tilde{Q}, \tilde{U}$ such that
$$\tilde{Q}\tilde{U}\tilde{Q}^* = \hat{A} \qquad \frac{\|\hat{A} - A\|}{\|A\|} = O(n^2 u)$$

- backward substitution on a triangular system $Ux = b$ computes $\tilde{x}$ s.t. $\hat{U}\tilde{x} = \hat{b}$, $\frac{\|\hat{U} - U\|}{\|U\|}, \frac{\|\hat{b} - b\|}{\|b\|} = O(n^2 u)$

Since the Bartels-Stewart algorithm is essentially Schur form + backward substitution, we can prove something:

$$U = I \otimes U_A + U_B \otimes I$$

$U \operatorname{vec}(Y) = \operatorname{vec}(E)$ & B-S solves this system by backsubstitution

The computed $\tilde{Y}$ satisfies

$$\hat{U}\, vec(\tilde{Y}) = vec(\hat{E}) \quad \text{with} \quad \frac{\|\hat{U}-U\|}{\|U\|}, \frac{\|\hat{E}-E\|}{\|E\|} = O(m^2n^2u)$$

With some triangle inequality arguments, one can obtain also
+orthogonality

the computed $\tilde{X}$ solves

$$\hat{K}\, vec(\tilde{X}) = vec(\hat{C}) \qquad \frac{\|\hat{K}-K\|}{\|K\|}, \frac{\|\hat{C}-C\|}{\|C\|} = O(m^2n^2u)$$

This relation can be used to prove that the residual of $\tilde{X}$ is small: let $\Delta_K = \hat{K}-K$, $\Delta_C = \hat{C}-C$

$$(K+\Delta_K)\, vec(\tilde{X}) = vec(C) + vec(\Delta_C)$$

$$\|K\, vec(\tilde{X}) - vec(C)\| = \|\Delta_K\, vec(\tilde{X}) + vec\, \Delta_C\|$$

$$\leq O(m^2n^2u)\left(\|K\|\,\|vec\, \tilde{X}\| + \|vec\, C\|\right)$$

$$\leq O(m^2n^2u)\left((\|A\|+\|B\|)\|vec(\tilde{X})\| + \|vec\, C\|\right)$$

The LHS is $\|K\, vec(\tilde{X}) - vec(C)\| = \|vec(A\tilde{X} - \tilde{X}B - C)\|$

$$= \|A\tilde{X} - \tilde{X}B - C\|_F$$

$\Rightarrow$ the relative residual

$$\frac{\|A\tilde{X} - \tilde{X}B - C\|_F}{(\|A\|+\|B\|)\|X\|_F + \|C\|_F}$$

is <u>always</u> of the order of machine precision for the Bartels-Stewart algorithm.

⚠ It is true that $\tilde{X}$ satisfies $\hat{K}\, vec(\tilde{X}) = vec(\hat{C})$

for $\hat{K}, \hat{C}$ close to K, C, but in general this does not ensure that $\hat{K} = I \otimes \hat{A} - \hat{B}^T \otimes I$ for suitable $\hat{A}, \hat{B}$.

The result $\tilde{X}$ is backward stable as a solution of $K \operatorname{vec}(X) = \operatorname{vec}(C)$, but not as a matrix equation in general; there are examples in which there are no $\hat{A}, \hat{B}, \hat{C}$ s.t. $\hat{A}\tilde{X} - \tilde{X}\hat{B} = \hat{C}$, and $\hat{A}, \hat{B}, \hat{C}$ are close to A, B, C.

B-S ensures that $\tilde{X}$ solves $\hat{K} \operatorname{vec} \tilde{X} = \operatorname{vec} \hat{C}$ exactly, with $\hat{K}, \hat{C}$ close to K, C

$$K \operatorname{vec} X = \operatorname{vec} C$$

⚠ If the problem is ill-conditioned, $\|\tilde{X} - X\|$ may be large!