Note Title 2025-03-07

The <u>invariant subspace</u> associated to $S \subseteq \mathbb{C}$ of a matrix M is

$$\mathcal{V} = \sum_{\lambda \in \Lambda(M) \cap S} \mathcal{V}_\lambda \qquad \mathcal{V}_\lambda = \{ v \in \mathbb{C}^n : (M - \lambda I)^k v = 0 \}$$

In some cases, they are less sensitive to perturbations than individual eigenvectors

<u>EXAMPLE</u>:

$$M = \left[\begin{array}{cc|cc} 0.5 & 1 & 1 & 1 \\ 0 & 0.5+\varepsilon & 1 & 1 \\ \hline 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2+\varepsilon \end{array}\right] \qquad \begin{bmatrix} \lambda & 1 \\ 0 & \mu \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & \mu \end{bmatrix}$$

(top-left block: A; bottom-right block: B)

$$\mathrm{sep}(A, B) \approx 0.7$$

$$\lambda_1 = 0.5 \qquad \lambda_2 = 0.5 + \varepsilon$$

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad v_2 = \begin{bmatrix} \varepsilon^{-1} \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{or} \quad v_2 = \begin{bmatrix} 1 \\ \varepsilon \\ 0 \\ 0 \end{bmatrix}$$

$$\mathcal{V}_S = \mathrm{Span}\{v_1, v_2\} = \mathrm{Im} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

is much less sensitive to perturbation than the two eigenvectors that span it.

$$\left[\begin{array}{c|ccc} 1-\varepsilon & 1 & 1 & 1 \\ \hline & 1+\varepsilon & 1 & 1 \\ & & 2 & 1 \\ & & & 2 \end{array}\right]$$

$$\mathrm{sep}(A,B) = \sigma_{min}\left(I \otimes A - B^T \otimes I\right)$$

Theorem: let (Stewart, Sun ~1960s)

$$M = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} \qquad V = \begin{bmatrix} I \\ 0 \end{bmatrix} \qquad \Lambda(A) \cap \Lambda(B) = \emptyset$$

and

$$M + \delta_M \qquad \delta_M = \begin{bmatrix} \delta_A & \delta_C \\ \delta_D & \delta_B \end{bmatrix}$$

$$a = \|\delta_A\| \quad , \quad b = \|\delta_B\| , \quad \dots$$

If

$$\left(\mathrm{sep}(A,B) - a - b\right)^2 - 4d\left(\|C\|_F + c\right) \geq 0 \qquad \beta^2 - 4\alpha\gamma \geq 0$$

$$\mathrm{sep}(A,B) - a - b \geq 0 \qquad \beta > 0$$

Then, there is a unique matrix X with

$$\|X\|_F \leq \frac{2d}{\mathrm{sep}(A,B) - a - b} \qquad \frac{2\alpha}{\beta}$$

such that $\begin{bmatrix} I \\ X \end{bmatrix}$ spans an invariant subspace of $M + \delta_M$.

Proof: We would like to find a triangular factorization

$$\begin{bmatrix} I & 0 \\ -X & I \end{bmatrix} \begin{bmatrix} A+\delta_A & C+\delta_C \\ \delta_D & B+\delta_B \end{bmatrix} \begin{bmatrix} I & 0 \\ X & I \end{bmatrix} = \begin{bmatrix} \hat{A} & \hat{C} \\ 0 & \hat{B} \end{bmatrix}$$

If we find an X that does this, then $\begin{bmatrix} I \\ X \end{bmatrix}$ spans the invariant subspace associated to the eigenvalues $\Lambda(\hat{A})$

The (2,1) block of the LHS is

$$\begin{bmatrix} -X & I \end{bmatrix} \begin{bmatrix} A+\delta_A & C+\delta_C \\ \delta_D & B+\delta_B \end{bmatrix} \begin{bmatrix} I \\ X \end{bmatrix} =$$

$$\begin{bmatrix} -X(A+\delta_A)+\delta_D & -X(C+\delta_C)+B+\delta_B \end{bmatrix} \begin{bmatrix} I \\ X \end{bmatrix}$$

$$= -X(A+\delta_A)+\delta_D - X(C+\delta_C)X + (B+\delta_B)X \overset{!}{=} 0$$

This is a nonlinear matrix equation, known as

algebraic Riccati equation

The linear part of this equation, $-X(A+\delta_A)+(B+\delta_B)X$ can be associated to a matrix using vectorization

$$\hat{K} = I \otimes (B+\delta_B) - (A+\delta_A)^T \otimes I$$

$$K = I \otimes B - A^T \otimes I$$

$$\sigma_{min}(K) = \text{sep}(B,A) = \text{sep}(A,B)$$

$$\sigma_{min}(\hat{K}) = \sigma_{min}\left(K + (I \otimes \delta_B - \delta_A^T \otimes I)\right)$$

Recall that for any matrices M, E, $\sigma_{min}(M+E) \geq \sigma_{min}(M) - \|E\|$,

so $\sigma_{min}(\hat{K}) \geq \sigma_{min}(K) - \|I \otimes \delta_B - \delta_A^T \otimes I\|$

$$\geq \text{sep}(A,B) - (a+b)$$

We can rewrite the Riccati equation as

$$\mathrm{vec}\left(X(A+\delta_A)-(B+\delta_B)X\right)=\mathrm{vec}\left(\delta_D - X(C+\delta_C)X\right)$$

$$\hat{K}\cdot \mathrm{vec}(X) = \mathrm{vec}\left(\delta_D - X(C+\delta_C)X\right)$$

$$\mathrm{vec}\, X = \underbrace{\hat{K}^{-1}\left(\delta_D - X(C+\delta_C)X\right)}_{\Phi(\mathrm{vec}(X))} \qquad \Phi : \mathbb{C}^{n^2} \to \mathbb{C}^{n^2}$$

A solution of the equation exists if and only if Φ has a fixed point.

We would like to use Brouwer's theorem to prove that one exists. We will show that there is r such that Φ sends a ball

$$\mathcal{B} = \{ x \in \mathbb{C}^{n^2} : \|x\| \leq r \} \quad \text{to itself}$$

$$\mathcal{B} = \{ X \in \mathbb{C}^{n\times n} : \|X\|_F \leq r \}$$

Let us assume that $\|X\|_F \leq r$, then

$$\|\Phi(\mathrm{vec}(X))\| = \|\hat{K}^{-1} \mathrm{vec}\left(\delta_D + X(C+\delta_C)X\right)\|$$

$$\leq \|\hat{K}^{-1}\| \cdot \left(\|\delta_D\|_F + \|X\|_F \|C+\delta_C\|_F \|X\|_F \right)$$

$$\leq \frac{1}{\mathrm{sep}(A,B)-a-b}\left(d + (\|C\|_F + c)r^2\right)$$

If we find $r>0$ such that $\frac{1}{\mathrm{sep}(A,B)-a-b}\left(d+(\|C\|_F+c)r^2\right) = r$,

then $\|\Phi(\mathrm{vec}(X))\| \leq r$, so Φ sends $\mathcal{B}$ to itself.

$$\beta = \mathrm{sep}(A,B) - a - b, \quad \alpha = d, \quad \gamma = \|C\|_F + c$$

$$\frac{1}{\beta}\left(\alpha+\gamma r^2\right)=r \iff \alpha-\beta r+\gamma r^2=0$$

We switch to the equation $\alpha s^2-\beta s+\gamma=0$

with solutions $s_\pm=\frac{1}{r_\mp}$

$$s_\pm=\frac{\beta\pm\sqrt{\beta^2-4\alpha\gamma}}{2\alpha}$$

if $\boxed{\beta^2-4\alpha\gamma\geq 0}$, we have two real solutions.

We assume $\alpha,\beta>0$, so that the larger of the two solutions is

$$s_+=\frac{\beta+\sqrt{\beta^2-4\alpha\gamma}}{2\alpha}\geq\boxed{\frac{\beta}{2\alpha}}$$

I.e. there is

$0<r_-\leq\frac{2\alpha}{\beta}$ that solves $\alpha-\beta r+\gamma r^2=0$.

We found r such that $\mathcal{B}=\{x:\|x\|\leq r\}$ is sent to itself by Φ, then there is a solution to $\Phi(x)=x$ with $\|X\|_F\leq r$

X solves the algebraic Riccati equation, hence $\begin{bmatrix}I\\X\end{bmatrix}$ is an invariant subspace of $M+\delta_M$ $\square$

$$\hat{A}=A+\underset{\sim}{\delta_A}+(C+\underset{\sim}{\delta_C})\underset{\sim}{X}$$

$$\|\hat{A}-A\|_F\leq a+(\|C\|_F+c)\frac{2d}{\operatorname{sep}(A,B)-a-b}$$

Matrix functions: given $f: U \subset \mathbb{C} \to \mathbb{C}$,

can we generalize it to matrices $A \in \mathbb{C}^{n\times n}$?

E.g. $\exp(A) = I + A + \frac{A^2}{2} + \frac{A^3}{3!} + \dots$

To give a definition, we start studying simple functions and simple matrices: $J \in \mathbb{C}^{k\times k}$ Jordan block with 0

$$J = \begin{bmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix} \qquad J^2 = \begin{bmatrix} 0 & 0 & 1 & & \\ & 0 & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & \ddots & 0 \\ & & & & 0 \end{bmatrix} \qquad J^3 = \begin{bmatrix} 0 & 0 & 0 & 1 & \\ & \ddots & & \ddots & \ddots \\ & & \ddots & & 1 \\ & & & \ddots & 0 \\ & & & & 0 \\ & & & & 0 \end{bmatrix}$$

$$\dots \quad J^{k-1} = \begin{bmatrix} 0 & \cdots & 0 & 1 \\ \vdots & & & 0 \\ & & & \vdots \\ 0 & \cdots & & 0 \end{bmatrix}, \quad J^k = 0$$

Given a polynomial

$$p(x) = \sum_{i=0}^{d} c_i x^i,$$

$$p(J) = \sum_{i=0}^{d} c_i J^i = \begin{bmatrix} c_0 & c_1 & c_2 & \cdots & c_{k-1} \\ & \ddots & \ddots & \ddots & \vdots \\ & & \ddots & \ddots & c_2 \\ & 0 & & \ddots & c_1 \\ & & & & c_0 \end{bmatrix}.$$

Now let us take a general eigenvalue λ, and

$$J_\lambda = \begin{bmatrix} \lambda & 1 & & 0 \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{bmatrix}.$$ What is $p(J_\lambda)$

We can write

$$p(x) = p(\lambda) + p'(\lambda)(x-\lambda) + \frac{p''(\lambda)}{2}(x-\lambda)^2 + \dots + \frac{p^{(d)}(\lambda)}{d!}(x-\lambda)^d$$

(no remainder, since $p^{(k)}(x) = 0$ for $k > d$)

$$p(J_\lambda) = p(\lambda) I + p'(\lambda)(J_\lambda - \lambda I) + \frac{p''(\lambda)}{2}(J_\lambda - \lambda I)^2 + \dots$$

$$\dots + \frac{p^{(d)}(\lambda)}{d!}(J_\lambda - \lambda I)^d$$

Note that $J_\lambda - \lambda I = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & & 1 \\ & & & 0 \end{bmatrix}$

$$p(J_\lambda) = \begin{bmatrix} p(\lambda) & p'(\lambda) & \cdots & \frac{p^{(k-1)}(\lambda)}{(k-1)!} \\ & \ddots & \ddots & \vdots \\ & 0 & & p(\lambda) \end{bmatrix}$$

Let now A be a matrix with Jordan decomposition

$$A = V J V^{-1}$$

$$J = \text{diag}(J_{\lambda_1}, J_{\lambda_2}, \dots J_{\lambda_s}) \qquad J_{\lambda_i} \in \mathbb{C}^{k_i \times k_i}$$

Using this decomposition, we can write

$$p(A) = V p(J) V^{-1} = V \,\text{diag}\left(p(J_{\lambda_1}), p(J_{\lambda_2}), \dots p(J_{\lambda_s})\right) V^{-1}$$

$$p(J_{\lambda_i}) = \begin{bmatrix} p(\lambda_i) & p'(\lambda_i) & \cdots & \frac{p^{(k_i-1)}(\lambda_i)}{(k_i-1)!} \\ & \ddots & \ddots & \vdots \\ & 0 & & p(\lambda_i) \end{bmatrix}$$

<u>Definition</u>: Given a matrix A with Jordan blocks $J_{\lambda_i} \in \mathbb{C}^{k_i \times k_i}$, $i = 1, \dots s$, and given a function f

that is differentiable at least $k_i - 1$ times in λ_i

we say that f <u>is defined</u> on A, and

$$f(A) := V \operatorname{diag}\left(f(J_{\lambda_1}), \dots, f(J_{\lambda_s})\right) V^{-1}$$

$$f(J_{\lambda_i}) = \begin{bmatrix} f(\lambda_i) & f'(\lambda_i) & \cdots & \frac{f^{(k_i-1)}(\lambda_i)}{(k_i-1)!} \\ & \ddots & \ddots & \vdots \\ 0 & & & \cdots \end{bmatrix}.$$

<u>Problem:</u> V is not unique! Is this well-defined?

To prove this, we switch to a different alternative definition

Given f, A, let us construct a polynomial $p(x)$ such that

$$f(\lambda_i) = p(\lambda_i), \quad f'(\lambda_i) = p'(\lambda_i), \dots \quad f^{(k_i-1)}(\lambda_i) = p^{(k_i-1)}(\lambda_i) \qquad (*)$$

for all $i = 1, \dots s$

(Hermite interpolation)

and we define $f(A) := p(A) = \sum_{i=0}^{d} c_i A^i$

This definition does not depend on V from a Jordan decomposition, but it depends on p!

However, if the interpolation conditions $(*)$ hold, then

$$p(J_{\lambda_i}) = \begin{bmatrix} p(\lambda_i) & p'(\lambda_i) & \cdots & \frac{p^{(k_i-1)}(\lambda)}{(k_i-1)!} \\ & \ddots & \ddots & \ddots \\ 0 & & & \end{bmatrix}$$

is uniquely defined, and so is $p(A)$. More formally:

<u>Lemma</u>: given two polynomials p, q such that

$$p(\lambda_i) = q(\lambda_i), \ldots \quad p^{(k_i-1)}(\lambda_i) = q^{(k_i-1)}(\lambda_i)$$

$$p(A) = q(A)$$

<u>Proof</u>: from the expression of $p(VJV^{-1})$, once we fix <u>one</u> Jordan form of A. $\square$