Lyapunov equations

Note Title 2025-05-02

$$A^*W + WA + Q = 0 \qquad (L)$$

$A, Q \in \mathbb{C}^{n\times n}$ $\quad Q = Q^*$

$W \in \mathbb{C}^{n\times n}$ unknown

Special case of $AX - XB = C$

Unique solution if and only if $\Lambda(A^*) \cap \Lambda(-A) = \phi$

This holds in particular if $\Lambda(A) \subset LHP$ (open)

because in this case $\Lambda(A^*) \subset LHP$ $\Lambda(-A) \subset RHP$

Lemma suppose (L) has a unique solution W, then $W = W^*$

Proof: transpose (L), to obtain

$$A^*W + WA + Q = 0 \Rightarrow W^*A + A^*W^* + Q = 0$$

$\Rightarrow W^*$ also solves (L)

$\Rightarrow W^* = W$. $\square$

Lemma: suppose $\Lambda(A) \subset LHP$. Then, the solution W can be expressed as an integral

$$W = \int_0^\infty \exp(A^*t)\, Q \exp(At)\, dt.$$

(Note that the integral converges:

$$\exp(At) = V \cdot \begin{bmatrix} \exp(J_1 t) & & \\ & \ddots & \\ & & \exp(J_s t) \end{bmatrix} V^{-1}$$

$$\exp\left(\begin{bmatrix} \lambda & 1 & & \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{bmatrix} t\right) = \begin{bmatrix} e^{\lambda t} & e^{\lambda t} t & \cdots & \frac{1}{(k-1)!} e^{\lambda t} t^{k-1} \\ & \ddots & \ddots & \\ & & & \end{bmatrix}$$ → exponential decay)

Proof: differentiate

$$\frac{d}{dt} \exp(A^* t) Q \exp(At) = A^* \exp(A^* t) Q \exp(At) + \exp(A^*)t\, Q \exp(At) \cdot A$$

Integrate both sides:

$$\left[\exp(A^* t) Q \exp(At)\right]_0^\infty = A^* \int_0^\infty \exp(A^* t) Q \exp(At)\, dt + \int_0^\infty \exp(A^* t) Q \exp(At)\, dt\, A$$

$\infty \to 0$, $0 \to Q$

$$0 - Q \qquad = A^* W + WA$$

Lemma: suppose $\Lambda(A) \subset LHP$, and $Q \succeq 0$ (positive semidef.)

Then, $W \succeq 0$. (And $Q \succ 0 \Rightarrow W \succ 0$)

Proof:

$$W = \int_0^\infty \underbrace{\exp(A^* t) Q \exp(At)}_{\succeq 0} dt$$

The integrand is semidef $\Rightarrow W \succeq 0$.

Lemma: Suppose $Q \succ 0$, $W \succ 0$. Then, $\Lambda(A) \subset LHP$

Proof: $Ax = x\lambda$ eigenvector/value pair

$$x^* (A^* W + WA + Q) x = 0$$

$$\lambda^* x^* W x + x^* W x \lambda + x^* Q x = 0$$

$$2\operatorname{Re}(\lambda) = \lambda^* + \lambda = -\frac{x^* Q x}{x^* W x} < 0. \square$$

This is useful to prove that $\Lambda(A) \subset LHP$ without actually computing the eigenvalues

Def: a continuous-time linear dynamical system is the ODE

$$\begin{cases} x(0) = x_0 \\ \dot{x}(t) = Ax(t) \end{cases}$$

We know that

$$x(t) = \exp(At) \cdot x_0$$

and that

$$\lim_{t \to \infty} x(t) = 0 \text{ iff } \Lambda(A) \subset LHP$$

suppose you have $W \succ 0$, $Q \succ 0$ s.t. $A^*W + WA + Q = 0$

Original argument: define the energy function $V(x) = x^* W x$ for a given $W \succ 0$. Then, we show that $V(x(t))$ is decreasing:

$$\frac{d}{dt} V(x(t)) = \frac{d}{dt} x(t)^* W x(t) = x(t)^* A^* W x(t) + x(t)^* W A x(t)$$

$$= x(t)^* (A^*W + WA) x(t) = x(t)^* (-Q) x(t) < 0$$

One can prove that $V(x(t)) \to 0$, and this implies $x(t) \to 0$

This proves: if there are $W, Q \succ 0$ s.t. $A^*W + WA + Q = 0$, then $x(t) \to 0$.

There is an analogous theory for discrete-time dynamical systems: consider a sequence of vectors

$$\begin{cases} x_0 \in \mathbb{C}^n \\ x_{k+1} = A x_k \end{cases}$$

$$x_k = A^k x_0$$

$$\lim_{k \to \infty} x_k = 0 \text{ iff } \Lambda(A) \subset Disc$$

$$Disc = \{ z \in \mathbb{C} : |z| < 1 \}$$

Stein equation (discrete-time Lyapunov equation)

$$W - A^*WA = Q \qquad (S)$$

Expression for its solution:

$$W = \sum_{k=0}^{\infty} (A^*)^k Q A^k \quad \text{converging if} \quad \Lambda(A) \subset \text{Disc}$$

$$W = \sum_{k=0}^{\infty} (A^*)^k Q A^k = Q + A^* \left(\sum_{k=0}^{\infty} (A^*)^k Q A^k \right) A = Q + A^* W A$$

Given A, Q, W that satisfy (S):

- if $\Lambda(A) \subset \text{Disc}$, $Q > 0 \Rightarrow W > 0$: from the formula for W
- if $Q > 0$, $W > 0 \Rightarrow \Lambda(A)$ disc:

$$Ax = x\lambda$$

$$x^*(W - A^* W A)x = x^* Q x$$

$$x^* W x - \bar{\lambda} x^* W x \lambda = x^* Q x$$

$$(1 - \bar{\lambda}\lambda) = \frac{x^* Q x}{x^* W x} > 0$$

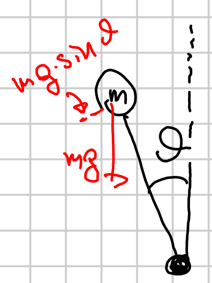
Given $W > 0$, $Q > 0$, A that satisfy (S),
Define $V(x) = x^* W x$. We show that $V(x_k)$ is decreasing:

$$V(x_{k+1}) - V(x_k) = x_{k+1}^* W x_{k+1} - x_k^* W x_k =$$

$$= x_k^* (A^* W A - W) x_k = - x_k^* Q x_k < 0$$

Control systems

inverted pendulum:



One degree of freedom, angle θ

Equations of motion:

$$\cancel{m} g \sin\theta = \cancel{m} \ddot{\theta}$$

$\ddot{\vartheta} = g \sin\vartheta \approx g\vartheta$ if ϑ is small

Define the state: $x(t) = \begin{bmatrix} \vartheta(t) \\ \dot{\vartheta}(t) \end{bmatrix}$

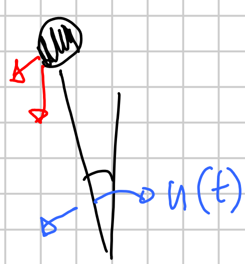
$$\dot{x}(t) = \begin{bmatrix} \dot{\vartheta}(t) \\ \ddot{\vartheta}(t) \end{bmatrix} = \begin{bmatrix} \dot{\vartheta}(t) \\ g\vartheta(t) \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ g & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} \vartheta(t) \\ \dot{\vartheta}(t) \end{bmatrix}}_{x(t)} = Ax(t)$$

$$\dot{x}(t) = Ax(t)$$

Is this system stable? I.e. if you start from $x(0) = x_0 \neq 0$, does the system converge to $\lim_{t \to \infty} x(t) = 0$?

$$\Lambda(A) = \text{zeros of } \det\begin{pmatrix} x & 1 \\ g & x \end{pmatrix} = \text{zeros of } x^2 - g = \{\pm\sqrt{g}\} \notin LHP$$

Let us add an additional force to keep the system stable:



$$\ddot{\vartheta}(t) = g \cdot \vartheta(t) + u(t)$$

$$x(t) = \begin{bmatrix} \vartheta(t) \\ \dot{\vartheta}(t) \end{bmatrix}$$

$$\dot{x}(t) = \begin{bmatrix} \dot{\vartheta}(t) \\ \ddot{\vartheta}(t) \end{bmatrix} = \begin{bmatrix} \dot{\vartheta}(t) \\ g\vartheta(t) + u(t) \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ g & 0 \end{bmatrix}}_{= A} \begin{bmatrix} \vartheta(t) \\ \dot{\vartheta}(t) \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{= B} u(t)$$

$$\dot{x}(t) = Ax(t) + Bu(t)$$

We can choose $u(t) = Fx(t) = [f_1 \ \ f_2] \begin{bmatrix} \vartheta(t) \\ \dot{\vartheta}(t) \end{bmatrix}$

If we add this control function to the system,

$$\dot{x}(t) = Ax(t) + BFx(t) = \left(\begin{bmatrix} 0 & 1 \\ g & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} [f_1 \ \ f_2] \right) x(t)$$

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ g+f_1 & f_2 \end{bmatrix} x(t)$$

$$\det\left(\begin{bmatrix} 0 & 1 \\ g+f_1 & f_2 \end{bmatrix} - xI \right) = x^2 - f_2 x - (f_1 + g)$$

If $f_2 = 0$, then $\det((A+BF) - x1) =$ zeros of $x^2 - (f_1 + g)$

$\not\subset$ LHP for any choice of f_1

$\Rightarrow$ just observing the <u>position</u> is not sufficient to <u>stabilize</u> the system.

General theory:

$\dot{x}(t) = Ax(t) + Bu(t)$ $\qquad x : [0, \infty) \to \mathbb{R}^n$ <u>state</u>

$\qquad u : [0, \infty) \to \mathbb{R}^m$ <u>control</u>

$A \in \mathbb{C}^{n \times n}$, $B \in \mathbb{C}^{n \times m}$

$m \leq n$

<u>Ex</u>: self-driving car, drone

<u>Ex</u>: heating control