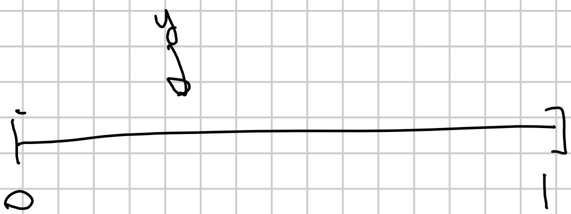
Control Theory

Note Title 2025-05-07

$$\dot{x} = Ax + Bu \qquad x : [0,\infty) \to \mathbb{R}^n \quad \text{state}$$

$$u : [0,\infty) \to \mathbb{R}^m \quad \text{control}$$

Heat control



Infinite-dim. version:

$x(y,t)$ ← time; y: position in $[0,1]$

$$\begin{cases} \dfrac{\partial x}{\partial t} = \alpha \dfrac{\partial^2 x}{\partial y^2} \\ x(1,t) = 0 \\ x(0,t) = u(t) \end{cases}$$

Infinite-dimensional A and B

$x_0(t) = u(t)$ $\quad x_1(t)$ $\quad \dots \quad$ $x_n(t)$ $\quad x_{n+1}(t) \equiv 0$

$$\dot{x}_k(t) = \alpha\Big(\big(x_{k+1}(t) - x_k(t)\big) + \big(x_{k-1}(t) - x_k(t)\big) \Big)$$

temp. differences to neighboring points

$$= \alpha\left(x_{k+1} - 2x_k + x_{k-1}\right)$$

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix} = \alpha \underbrace{\begin{bmatrix} -2 & 1 & & \\ 1 & -2 & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & -2 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \alpha \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{B} u(t)$$

$m = 1$

$$\dot{x} = Ax + Bu \qquad A \in \mathbb{R}^{n\times n},\ B \in \mathbb{R}^{n \times m}$$

Controllability: for which A, B can I choose $u(t)$ to get $x(t_F) = x_F$ for every prescribed (t_F, x_F)?

Stabilizability: for which A, B can I ensure $\lim_{t\to\infty} x(t) = 0$?

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u \qquad \begin{cases} \dot{x}_1(t) = A_{11}x_1(t) + A_{12}x_2(t) + B_1 u(t) \\ \dot{x}_2(t) = A_{22}x_2(t) \end{cases}$$

cannot control the second part $x_2(t)$

If $\Lambda(A_{22}) \subset LHP$, x_2 is stable, otherwise it's not, and control won't change it.

A change of variables might hide the zeros:

$\hat{x}(t) = Mx(t) \qquad M \in \mathbb{C}^{n\times n}$ invertible

$$\frac{d}{dt}\hat{x}(t) = MAM^{-1}\hat{x}(t) + MBu(t)$$

Change of variable in the state x: $(A, B) \to (MAM^{-1}, MB)$

$\square\ \square \to \square\ \square$

To analyze it, we need invariant subspaces

Lemma: the smallest subspace that is A-invariant and contains the columns of B

$$K(A,B) = \mathrm{Im}\,[B, AB, A^2B, A^3B, \ldots] \subseteq \mathbb{C}^n$$

Proof: It is A-invariant: indeed, if you take

$$v = \sum_{(i,j)\in S} A^i b_j c_{ij} \in K(A,B), \quad Av = \sum_{(i,j)} A^{i+1} b_j c_{ij} \in K(A,B)$$

It is the smallest: if a subspace contains the columns of B and it's A-invariant, then

$$\mathrm{Im}\,B \in \mathcal{U} \quad \mathrm{Im}\,AB \in \mathcal{U} \quad \mathrm{Im}\,A^2B \in \mathcal{U}, \ldots \qquad \square$$

If $B = b$ is one-dimensional, this is the "limit Krylov space"

$$\mathcal{U} = \bigcup K_n(A,b)$$

Definition: $K(A,B)$ is called the controllable space of the pair $(A,B) \in \mathbb{C}^{n\times n} \times \mathbb{C}^{n\times m}$.

A pair is called controllable if $K(A,B) = \mathbb{C}^n$.

Ex: $A \in \mathbb{C}^{n\times n}$, $B = [b]$ with b an eigenvector: $Ab = b\cdot\lambda$

$$K(A,B) = \mathrm{span}[b, b\lambda, b\lambda^2, \ldots] = \mathrm{span}(b) \text{ is one-dimensional}$$

We know from Krylov space theory that generically

$$[b, Ab, \ldots A^{n-1}b]$$

are linearly independent.

Remarks: • controllability only depends on $\mathrm{Im}(B)$, so

$$K(A,B) = K(A,BS) \text{ for every } S \in \mathbb{C}^{m\times m} \text{ invertible}$$

or $S \in \mathbb{C}^{m \times p}$ with full row rank.

- Also, $K(A-\alpha I, B) = K(A,B)$

$$(A-\alpha I)^j B = A^j B + \alpha j A^{j-1} B + \dots + \alpha^{j-1} j AB + \alpha^j B$$

<u>Theorem</u>: The following are equivalent:

1. The system $\dot{x} = Ax + Bu$ is controllable, i.e., for any $x_F, t_F,$ $(t_F > 0)$ one can choose $u(t)$ such that $x(t_F) = x_F$.

2. The pair (A,B) is controllable, i.e. $K(A,B) = \mathbb{C}^n$.

3. The matrix
$$W_{t_F} = \int_0^{t_F} \exp(At) B B^* \exp(A^* t)\, dt$$
is invertible (for one choice of t_F, or, equivalently, for <u>all</u> $t_F > 0$)

(Note that W_∞ is the solution of $AW + WA^* + BB^* = 0$)

<u>Proof</u>: $1 \Rightarrow 2$ $\quad$ not 2 $\Rightarrow$ not 1

$K(A,B)$ is <u>not</u> $\mathbb{C}^n$

$\dot{x} = Ax + \underbrace{Bu}_{f(t)}$ $\rightarrow$ from Calculus 2,

$$x(t_F) = \exp(At_F) x_0 + \int_0^{t_F} \exp(A(t_F - t)) \underbrace{B u(t)}_{f(t)}\, dt$$

$$x(t_F) - \exp(At_F) x_0 = \underbrace{\int_0^{t_F} \underbrace{\underbrace{\exp(A(t_F - t))}_{p(A)} B}_{\in K(A,B)} u(t)\, dt}_{\in K(A,B)}$$

$\Rightarrow x(t_F) - \exp(At_F)x_0 \in K(A,B)$ for all $u(t)$

$\to$ We can not get all target states $x_F = x(t_F)$, only those in $\exp(At_F)x_0 + K(A,B)$

$2 \Rightarrow 3$ not-3 $\Rightarrow$ not-2

$W_{t_F} = \int_0^{t_F} \exp(At)BB^* \exp(A^* t)\,dt$ is s.t. $W_{t_F} v = 0$

We want to prove that $K(A,B) \neq \mathbb{C}^n$

Note that $W_{t_F} \succeq 0$, because the integrand is psd.

So $W_{t_F} v = 0 \iff v^* W_{t_F} v = 0$

$$v^* W_{t_F} v = \int_0^{t_F} \|v^* \exp(At)B\|^2\,dt$$

If we want this integral to be 0, then

$$\phi(t) = v^* \exp(At)B$$

must be the identically-zero function.

$\phi(0) = 0 \Rightarrow v^* B = 0$

$\phi'(0) = 0 \Rightarrow v^* AB = 0$ $\Rightarrow v^* A^j B = 0$ for all j.

$\phi''(0) = 0 \Rightarrow v^* A^2 B = 0$

$\vdots$

So $K(A,B) \in v^\perp$

$3 \Rightarrow 1$

W_{t_F} invertible $\Rightarrow$ the system is controllable. $\dot{x} = Ax + Bu$

We will exhibit a control s.t. $x(t_F) = x_F$:

$\underline{u(t) = B^* \exp(A^*(t_F - t))y}$ for a certain $y \in \mathbb{C}^n$

We plug it into the formula for $x(t_F)$:

$$x(t_F) = \exp(At_F)x_0 + \int_0^{t_F} \underbrace{\exp(A(t_F-t))B}\,\underbrace{B^*\exp(A^*(t_F-t))}\, y\, dt$$

$$= \exp(At_F)x_0 + W_{t_F}\, y$$

Since W_{t_F} is invertible, we can choose

$$y = W_{t_F}^{-1}\left(x_F - \exp(At_F)x_0\right)$$

to get $x(t_F) = x_F$. $\square$

Assume $\Lambda(A) \subset LHP$

W_{t_F} is an increasing function of t_F

$W_{t_2} - W_{t_1} \geq 0$ if $t_2 \geq t_1$ so if $W_{t_1} \succ 0$ then $W_{t_2} \succ 0$

<u>Theorem</u>: if $\Lambda(A) \subset LHP$, then the solution of $AW + WA^* + BB^* = 0$ is pos def <u>if and only if</u> (A, B) controllable.

<u>Related results</u>: • If we make a change of basis with $M = [M_1\ M_2]$ and M_1 is a basis of $K(A,B)$ we get

$$M^{-1}AM = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \qquad M^{-1}B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}$$

with (A_{11}, B_1) controllable

(<u>Kalman decomposition</u>)

• controllability criterion: (A,B) controllable if and only if $\operatorname{rank}\,[A - \lambda I \mid B] = n$ for all $\lambda \in \mathbb{C}$

(Hautus/Popov test)

Controllability: $x(t_f) = x_f$ $\qquad$ $x(t_f) = 0$, and then $x(t) \equiv 0$

Stabilizability: $\lim x(t) = 0$

A system is stabilizable iff in the Kalman decomposition

$$A = M^{-1}\begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} M \qquad B = M^{-1}\begin{bmatrix} B_1 \\ 0 \end{bmatrix}$$

we have $\Lambda(A_{22}) \subset LHP$

Methods to test controllability numerically:

- compute $\operatorname{rank}[B, AB, \dots A^{n-1}B]$ $\quad$ (A^n is a lin. comb. of the earlier powers, by Cayley-Hamilton)
- If $B = [b]$, you can run Arnoldi and check for breakdown
- If $\Lambda(A) \subset LHP$, you can solve $AW + WA^* + BB^* = 0$, and check if $W \succ 0$

They all rely on a rank decision.
Like testing singularity, testing controllability is ill-posed, it depends on exact zeros.
There are methods to compute distance to uncontrollability:

$$d(A,B) = \min\left(\|\hat{A}-A\|^2 + \|\hat{B}-B\|^2 \text{ over } (\hat{A},\hat{B}) \text{ uncontrollable}\right)$$

(Datta, distance to uncontrollability chapter)

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sist. non controllabile $\iff \exists \lambda$ t.c. $\operatorname{rk}[A-\lambda I \;\; B] < n$

If I knew λ, then I have to find

Staircase decomposition

$$\sigma_{min}\left(\begin{bmatrix} A-\lambda I & B \end{bmatrix}\right)$$

$$n\underbrace{\begin{bmatrix} \quad & \Big| & \quad \end{bmatrix}}_{n+m}$$