Linear control systems

Note Title 2025-05-15

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ x_0 \text{ given} \end{cases} \qquad A \in \mathbb{C}^{n\times n} \quad B \in \mathbb{C}^{n\times m}$$

$$x : [0,\infty) \to \mathbb{R}^n$$
$$u : [0,\infty) \to \mathbb{R}^m$$

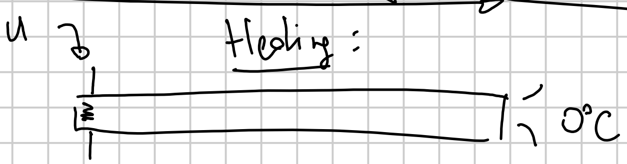
$F \in \mathbb{C}^{n\times m}$ $\quad u(t) = Fx(t)$ $\quad$ Feedback control

$$\dot{x} = (A+BF)x$$

$$A+BF = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} [f_1, f_2] = \begin{bmatrix} 0 & 1 \\ 1+f_1 & f_2 \end{bmatrix}$$

$$\det\left(\begin{bmatrix} -x & 1 \\ 1+f_1 & f_2 - x \end{bmatrix}\right) = x^2 - f_2 x - (1+f_1)$$

Heating:



$$\dot{x} = Ax + Bu \qquad A \begin{bmatrix} -2 & 1 & & 0 \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ 0 & & 1 & -2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Control s.t. $x(t_F) = x_F$.

$$W = \int_0^{t_F} \exp(tA) B B^* \exp(tA^*)\, dt$$

$$u(t) = B^* \exp((t_F - t)A^*)\, y \qquad y = W^{-1}(x_F - \exp(t_F A)x_0)$$

How to compute a stabilizing feedback, i.e. F s.t.

$$\Lambda(A+BF) \subset LHP$$

Theorem (Bass algorithm) (A,B) controllable, $\alpha > \rho(A)$, W solution of

$$(-\alpha I - A)W + W(-\alpha I - A)^* + \underbrace{2BB^*}_{Q} = 0.$$

Then, $W \succ 0$, and $F = -B^* W^{-1}$ is a stabilizing feedback.

Proof: Note that $\Lambda(-\alpha I - A) \subset LHP$, because of how we chose α.

$Q \succeq 0$ if $(-\alpha I - A, \sqrt{2}B)$ is controllable, then $W \succ 0$.

(Recall that $\hat{A}W + W\hat{A}^* + \hat{B}\hat{B}^* = 0$ has solution

$$W = \int_0^\infty \exp(t\hat{A})\hat{B}\hat{B}^* \exp(t\hat{A})\,dt$$

if $\Lambda(\hat{A}) \subset LHP$, and if $\hat{A}, \hat{B}$ is controllable, then $W \succ 0$.)

But

$$\operatorname{span}\left(\sqrt{2}B, (-\alpha I - A)\sqrt{2}B, (-\alpha I - A)^2\sqrt{2}B, \dots\right) = \mathbb{C}^n$$

$$= \operatorname{span}\left(B, AB, A^2B, \dots\right) = \mathbb{C}^n \text{ because } (A,B) \text{ controllable}$$

$$(*) \quad (-\alpha I - A)W + W(-\alpha I - A)^* + 2BB^* = 0 \qquad F = -B^* W^{-1}$$

We can see that

$$(A+BF)W + W(A+BF)^* + 2\alpha W$$

$$= (A + \alpha I)W + W(A^* + \alpha I) - BB^*W^{-1}W - WW^{-1}BB^*$$

$$= (\alpha I + A)W + W(\alpha I + A)^* - 2BB^* = 0 \text{ because of } (*)$$

Since $(A+BF)W + W(A+BF)^* + 2\alpha W = 0$,

the solution W and the coefficient matrix $2\alpha W$ are $\succ 0$

$\Rightarrow \Lambda(A+BF) \subset LHP$.

For a question:

Kalman decomposition

$M = [M_1 \;\; M_2]$

$$\mathrm{Im}\, M_1 = \mathrm{Im}[B, AB, A^2B, \dots]$$

$$A(\mathrm{Im}\, M_1) \subset \mathrm{Im}\, M_1 \quad (A\text{-invariante})$$

$$\Rightarrow \quad M^{-1}AM = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$$

$\mathrm{Im}\, B \subset \mathrm{Im}\, M_1$ $\qquad M^{-1}B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}$

(A_{11}, B_1) controllabile

$$M^{-1}\mathrm{Im}(B, AB, A^2B, \dots) = \mathrm{Im}\left[\begin{pmatrix} B_1 \\ 0 \end{pmatrix}, \begin{pmatrix} A_{11}B_1 \\ 0 \end{pmatrix}, \begin{pmatrix} A_{11}^2 B_1 \\ 0 \end{pmatrix}, \dots\right]$$

ha dimensione uguale al num. di colonne di M_1

$$\Rightarrow \dim(B_1, A_{11}B_1, A_{11}^2B_1, \dots) = \mathbb{C}^{n_1}$$

Optimal control

Among all stabilizing control functions $u(t)$ (piecewise continuous to fix a space), we look for the one that minimizes a certain cost function

$$V(u) = \int_0^\infty (x^* Q x + u^* R u)\, dt$$

$Q \in \mathbb{C}^{n \times n}$ $\quad Q = Q^*$, $\quad Q \succeq 0$

$R \in \mathbb{C}^{m \times m}$ $\quad R = R^*$, $\quad \boxed{R \succ 0}$

strictly pd

$$\boxed{\begin{array}{l} \min \int_0^\infty (x^* Q x + u^* R u)\, dt \\ \text{s.t. } \begin{cases} \dot{x} = Ax + Bu \\ x(0) = x_0 \end{cases}, \quad \lim_{t \to \infty} x(t) = 0 \\ \text{over all possible } u \end{array}}$$

Theorem: Let $Q \succcurlyeq 0$, $R \succ 0$, (A,B) controllable, (A^*, Q) controllable, $x_0 \in \mathbb{C}^n$ given

Set $G = BR^{-1}B^*$ $\quad G \in \mathbb{C}^{n\times n}$, $G \succcurlyeq 0$

Then, there exists $X = X^* \in \mathbb{C}^{n\times n}$ that satisfies

$$\boxed{A^*X + XA + Q - XGX = 0}$$ (Algebraic Riccati equation)

and $\Lambda(A - GX) \subset LHP$

(note that $A - GX = A - BR^{-1}B^*X = A + BF$ with $F = -R^{-1}B^*X$).

The optimal value of the minimum problem that we stated above is $x_0^* X x_0$, and the optimal $u(t)$ is $u(t) = -R^{-1}B^*X x(t) = Fx(t)$.

Note that the optimal control is a feedback control, and that to compute F we need to find X numerically.

X is a solution of ARE and it is stabilizing, $\Lambda(A-GX) \subset LHP$

We start by assuming that X satisfying that equation exists, and show that $u(t) = -R^{-1}B^*Xx(t)$ is the solution of that optimization problem.

$$\frac{d}{dt} x(t)^* X x(t) = \dot{x}(t)^* X x(t) + x(t)^* X \dot{x}(t)$$

$$= (Ax + Bu)^* X x + x^* X (Ax + Bu)$$

$$= x^*(A^*X + XA)x + u^*B^*Xx + x^*XBu$$

$$= x^*(XBR^{-1}B^*X - Q)x + u^*B^*Xx + x^*XBu \quad \begin{matrix} +u^*Ru \\ -u^*Ru \end{matrix}$$

$$= (u + R^{-1}B^*Xx)^* R (u + R^{-1}B^*Xx) - x^*Qx - u^*Ru \geq -x^*Qx - u^*Ru$$

This manipulation holds for all choices of $u(t)$ and their associated $x(t)$.

$$\int_0^\infty (x^* Q x + u^* R u)\, dt \overset{\forall}{\geq} -\int_0^\infty \frac{d}{dt} x(t)^* X x(t)$$

$$= \Big[-x(t)^* X x(t) \Big]_0^\infty = x_0^* X x_0 - \underbrace{x(\infty)^* X x(\infty)}_{\overset{\shortparallel}{0} \text{ because stabilizing}}$$

So the integral is at least $x_0^* X x_0$, and these become equalities if $u(t) + R^{-1} B^* X x(t) = 0$ everywhere.

i.e. $u(t) = F x(t)$.