Algebraic Riccati equations

Note Title 2025-05-16

To compute optimal control, we need to solve

$$A^* X + XA + Q - XGX = 0 \qquad \text{Algebraic Riccati equation}$$

$$G = BR^{-1}B^* \qquad Q \succcurlyeq 0 \qquad R \succ 0$$

$$A, X, Q, G \in \mathbb{C}^{n \times n}$$

We need X s.t. $\Lambda(A - GX) \subset LHP$, $X = X^*$

(We also need to prove that such a solution exists unique)

<u>Assumptions</u>: (A, B) and (A^*, Q) are controllable

(A, B) controllable $\Leftrightarrow$ (A, G) controllable

because controllable $\Leftrightarrow$ $(\mathrm{rk}\,[B, AB, A^2B, \dots] = n)$

and B, G have same column space.

<u>Observation</u>: Riccati equation $\Leftrightarrow$

$$\begin{bmatrix} A & -G \\ -Q & -A^* \end{bmatrix} \begin{bmatrix} I_n \\ X \end{bmatrix} = \begin{bmatrix} I_n \\ X \end{bmatrix} (A - GX) \Leftrightarrow \begin{cases} A - GX = A - GX \\ -Q - A^*X = X(A - GX) \end{cases}$$

i.e. $\begin{bmatrix} I \\ X \end{bmatrix}$ is H-invariant, where $H = \begin{bmatrix} A & -G \\ -Q & -A^* \end{bmatrix} \in \mathbb{C}^{2n \times 2n}$

(the <u>Hamiltonian</u> of the system)

<u>Lemma</u>: Suppose $\mathrm{Im}\, U$, $U = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \in \mathbb{C}^{2n \times n}$ is H-invariant, and $U_1 \in \mathbb{C}^{n \times n}$ is invertible. Then, $X = U_2 U_1^{-1}$ solves the algebraic Riccati equation. If $\mathrm{Im}\, U$ is

associated to eigenvalues $\{\lambda_1, \lambda_2, \dots \lambda_n\}$, then

$$\Lambda(A - GX) = \{\lambda_1, \lambda_2, \dots \lambda_n\}.$$

Proof:

$$\underset{2n\times 2n}{\begin{bmatrix} A & -G \\ -Q & -A^* \end{bmatrix}} \underset{2n\times n}{\begin{bmatrix} U_1 \\ U_2 \end{bmatrix}} = \underset{2n\times n}{\begin{bmatrix} U_1 \\ U_2 \end{bmatrix}} \underset{n\times n}{S} \iff \begin{bmatrix} A & -G \\ -Q & -A^* \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} U_1^{-1} = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} U_1^{-1} U_1 S U_1^{-1}$$

$$\iff \begin{bmatrix} A & -G \\ -Q & -A^* \end{bmatrix} \begin{bmatrix} I \\ X \end{bmatrix} = \begin{bmatrix} I \\ X \end{bmatrix} U_1 S U_1^{-1}$$

$$\iff \begin{cases} A - GX = U_1 S U_1^{-1} \\ \underline{-Q - A^* X} = X U_1 S U_1^{-1} = \underline{X(A - GX)} \end{cases}$$

Underlined in red is the ARE.

$$\Lambda(A - GX) = \Lambda(S) = \{\lambda_1, \lambda_2, \dots \lambda_n\} \qquad \square$$

To understand existence of solutions, we study the matrix H

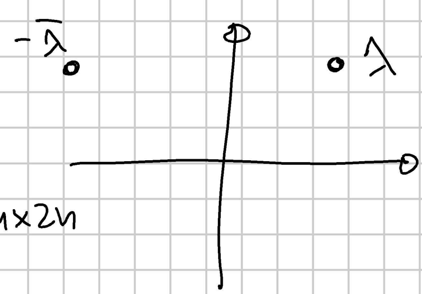
Def: A Hamiltonian matrix is one of the form

$$\begin{bmatrix} A & -G \\ -Q & -A^* \end{bmatrix}, \quad \text{with } G = G^*, \; Q = Q^*$$

$$\text{Hamiltonian matrices} = \left\{ \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \in \mathbb{C}^{2n\times 2n} : H_{11} = -H_{22}^*, \; H_{12} = H_{12}^*, \; H_{21} = H_{21}^* \right\}$$

Lemma: Let H be Hamiltonian, $\lambda \in \Lambda(H)$. Then, $-\bar{\lambda} \in \Lambda(H)$, and the two have the same algebraic multiplicity.

I.e. $\Lambda(H)$ is symmetric w.r.t. the imaginary axis



Proof: Let $J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} \in \mathbb{C}^{2n\times 2n}$

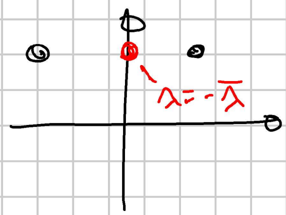
$$J^{-1}HJ = \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix}\begin{bmatrix} A & -G \\ -Q & -A^* \end{bmatrix}\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} =$$

$$= \begin{bmatrix} Q & A^* \\ A & -G \end{bmatrix}\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} = \begin{bmatrix} -A^* & Q \\ G & A \end{bmatrix} = -H^*$$

So $\Lambda(H) = \Lambda(-H^*)$ $\quad \lambda \in \Lambda(H) \Rightarrow \Lambda(-H^*) \Rightarrow -\bar{\lambda} \in \Lambda(H)$

$\square$

So the spectrum of H is symmetric w.r.t imaginary axis.



If there are no imaginary eigenvalues, then $\Lambda(H)$ has n eigenvalues in LHP and n in RHP

Theorem: under our assumptions $\left(Q \succeq 0 \quad G \succeq 0 \quad \begin{matrix} (A, G) \\ (A^*, Q) \end{matrix} \text{ contr.} \right)$

then H has no purely imaginary eigenvalues.

Proof: suppose by contradiction

$$\begin{bmatrix} A & -G \\ -Q & -A^* \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} i\omega \qquad \omega \in \mathbb{R}$$

$$\Leftrightarrow \begin{cases} (A - i\omega I) z_1 = G z_2 \\ -(A - i\omega I)^* z_2 = Q z_1 \end{cases} \Rightarrow \begin{cases} z_2^* (A - i\omega I) z_1 = z_2^* G z_2 \\ -z_1^* (A - i\omega I)^* z_2 = z_1^* Q z_1 \end{cases}$$

real since $G, Q \succeq 0$

$$\underbrace{z_1^* Q z_1}_{\substack{\geq \\ 0}} + \underbrace{z_2^* G z_2}_{\substack{\geq \\ 0}} = 0 \Rightarrow Q z_1 = G z_2 = 0$$

$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$ eigenvector $\Rightarrow$ z_1, z_2 not both zero.

If $z_1 \neq 0$, then we can write

$$z_i^* \left[(A - i\omega I)^* \quad Q \right] = 0 \qquad z_i^* \left[A^* + i\omega I \quad Q \right] = 0$$

$\Rightarrow$ (A^*, Q) not controllable because $z_1 (A^*)^* Q = 0$

This contradicts controllability (A^*, Q)

If instead $z_2 \neq 0$, then we get $z_2^* \left[A - i\omega I \quad G \right] = 0$

This contradicts controllability of (A, G). $\square$

Together, these results imply that H has n eigenvalues in the LHP and n in the RHP, counted with algebraic multiplicity

$\Rightarrow$ $\exists$ the (unique) invariant subspace of H associated to the LHP, i.e. $\exists! \; \mathcal{U} = \operatorname{Im} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$ such that

$$H \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} S \quad , \quad \Lambda(S) \in \text{LHP}$$

$\mathcal{U}$ is called <u>stable invariant subspace</u>.

We still need to prove that U_1 is invertible for this subspace.

<u>Theorem</u>: Under our assumptions, U_1 is invertible

<u>Proof</u>: in the notes, not too illuminating. $\square$

Properties of the subspace $\mathcal{U} = \operatorname{Im} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$.

<u>Lemma</u>: let $\begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$ be a basis matrix for the stable

invariant subspace of a Hamiltonian matrix, then

$$\begin{bmatrix} U_1^* & U_2^* \end{bmatrix} J \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = 0 \qquad \Longleftrightarrow \quad v^* J v = 0 \text{ for all } v \in \mathcal{U}$$

$n \times 2n \qquad 2n \times 2n \qquad 2n \times n$

$$\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$$

<u>Proof</u>: Let

$$H = V \begin{bmatrix} J_1 & \\ & J_2 \end{bmatrix} V^{-1}$$

be a Jordan form,

where $\Lambda(J_1) \subset LHP$, $\Lambda(J_2) \subset RHP$ $\quad J_1, J_2 \in \mathbb{C}^{n \times n}$

If $V = \begin{bmatrix} V_1 & V_2 \end{bmatrix} 2n$, then $\mathcal{U} = \operatorname{Im} V_1$, so $\begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$ and V_1
$\quad\quad\;\; n \quad n$

span the same subspace.

$$J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$\cancel{-H^* = V^{-*} \begin{bmatrix} -J_1^* & \\ & -J_2^* \end{bmatrix} V^*} \qquad -H^* = J H J^{-1} \qquad \begin{aligned} J^{-1} &= -J \\ J^* &= -J \end{aligned}$$

We want to show that JV_1 is the left invariant subspace for H associated to the unstable eigenvalues

$$V_1^* J^* H = (-J_1^*) V_1^* J^* \qquad J^* H = -JH = H^* J$$

$$V_1^* J^* H = V_1^* H^* J = (HV_1)^* J = (V_1 J_1)^* J = J_1^* V_1^* J = \underbrace{(-J_1)^*}_{RHP} V_1^* J^*$$

$\Rightarrow (JV_1)^*$ spans the <u>left</u> invariant subspace for H associated to its eigenvalues in the RHP

V_1 and JV_1 are orthogonal because they are the <u>left</u> and <u>right</u> invariant subspaces associated to disjoint eigenvalues

$$H = \begin{bmatrix} V_1 & V_2 \end{bmatrix} \begin{bmatrix} J_1 & \\ & J_2 \end{bmatrix} \begin{bmatrix} V_1 & V_2 \end{bmatrix}^{-1}$$

Im V_1 = Im first n columns of V
is the inv. subspace associated to eigenvalues in J_1

Im of last n rows of $W = V^{-1}$ is the inv. subspace associated to eigenvalues in J_2

$$V^{-1} = W = \begin{bmatrix} W_1 \\ \hline W_2 \end{bmatrix} \qquad V = \begin{bmatrix} V_1 & V_2 \end{bmatrix} \qquad W_2 \cdot V_1 = 0$$

$\square$

$$\Rightarrow 0 = \begin{bmatrix} U_1^* & U_2^* \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = U_1^* U_2 - U_2^* U_1 = 0$$

With this, we can prove symmetry of $X = U_2 U_1^{-1}$:

$$X^* - X = U_1^{-*} U_2^* - U_2 U_1^{-1} = U_1^{-*}\left(U_2^* U_1 - U_1^* U_2\right) U_1^{-1} = 0.$$

$\Rightarrow X$ is symmetric.

One way to compute invariant subspaces: through the reordered Schur form.

$$\begin{bmatrix} 1 & 0 \\ -X & 1 \end{bmatrix} H \begin{bmatrix} 1 & 0 \\ X & 1 \end{bmatrix} = \begin{bmatrix} \underbrace{\boxed{A - GX}}_{\in LHP} & -G \\ 0 & \underbrace{\boxed{-(A-GX)^*}}_{\in RHP} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ -X & 1 \end{bmatrix} \begin{bmatrix} A & -G \\ -Q & -A^* \end{bmatrix} \begin{bmatrix} 1 & 0 \\ X & 1 \end{bmatrix} = \begin{bmatrix} A & -G \\ -XA - Q & +XG - A^* \end{bmatrix} \begin{bmatrix} 1 & 0 \\ X & 1 \end{bmatrix}$$

$$= \begin{bmatrix} A - GX & -G \\ \underbrace{\boxed{-XA - Q + XGX - A^*X}}_{=0} & -(A-GX)^* \end{bmatrix}.$$