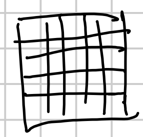
Large-scale matrix equations

Note Title 2025-05-23

Usually, large-scale control systems have:

- large, sparse $A \in \mathbb{C}^{n \times n}$ (from discretization)
- $B \in \mathbb{C}^{n \times m}$ tall thin, $m \ll n$

  few controllable degrees of freedom



Simplest possible large-scale equation: Lyapunov equation

$$AX + XA^* + \boxed{bb^*} = 0 \qquad b \in \mathbb{C}^n$$

$\leftarrow Q$ has rank-1

If you know how to solve equations with rank-1 rhs, you can also solve

$$AX + XA^* + b_1 b_1^* + b_2 b_2^* = 0$$

has solution $X_1 + X_2$, where
$$AX_1 + X_1 A^* + b_1 b_1^* = 0,$$
$$AX_2 + X_2 A^* + b_2 b_2^* = 0.$$

Also, if we know how to solve Lyapunov equations, we can solve large-scale Riccati equations with the Newton method.

We shall also assume $\Lambda(A) \subset LHP$.

<u>Problem</u>: solution X is full, in general

<u>Solution</u>: In most cases, X is well-approximated by a low-rank matrix, because it has rapidly decaying sing. values.

$$X \approx ZZ^* \qquad Z \in \mathbb{C}^{n \times k} \qquad k \ll n.$$

Our algorithms will compute Z, not X.

Idea: transform the Lyapunov equation into a (discrete-time) Stein equation

(LE) $$AX+XA^*+bb^*=0 \;\Rightarrow\; X-\hat{A}X\hat{A}^*=\hat{b}\hat{b}^*$$ (SE)

Theorem: let $\tau>0$, A with $\Lambda(A)\subset LHP$. Then, X solves the LE if and only if it solves the Stein equation SE with

$$\hat{A}=(A-\tau I)^{-1}(A+\tau I)\quad,\quad \hat{b}=\sqrt{2\tau}\,(A-\tau I)^{-1}b$$

And $\rho(\hat{A})<1$.

Proof: note that $\Lambda(A-\tau I)\subset LHP$.

$$X-(A-\tau I)^{-1}(A+\tau I)X(A+\tau I)^*(A-\tau I)^{-*}=2\tau(A-\tau I)^{-1}bb^*(A-\tau I)^{-*}$$

$$(A-\tau I)X(A-\tau I)^*-(A+\tau I)X(A+\tau I)^*=2\tau bb^* \iff$$

$$\cancel{AXA^*}-\tau XA^*-AX\tau+\cancel{\tau^2X}-\cancel{AXA^*}-\tau XA^*-AX\tau-\cancel{\tau^2X}=2\tau bb^*$$

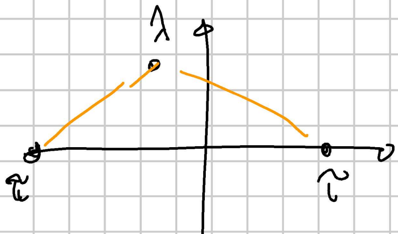
$$\iff 2\tau(AX+XA^*+bb^*)=0.$$

If $\lambda\in\Lambda(A)$, the corresponding eigenvalue of $\hat{A}$ is $\frac{\lambda+\tau}{\lambda-\tau}$

$$\Lambda(\hat{A})=\left\{\frac{\lambda+\tau}{\lambda-\tau},\ \text{where } \lambda\in\Lambda(A)\right\}$$

Since $\lambda\in LHP$,

$\text{dist}(\lambda,-\tau)<\text{dist}(\lambda,\tau)$



i.e. $\frac{|\lambda+\tau|}{|\lambda-\tau|}<1$, hence $\rho(\hat{A})<1$. $\square$

$$X-\hat{A}X\hat{A}^*=\hat{b}\hat{b}^* \qquad \rho(\hat{A})<1$$

We solve it via

$$\begin{cases} X_0 = 0 \\ X_{k+1} = \hat{b}\hat{b}^* + \hat{A} X_k \hat{A}^* \end{cases} \quad \Big| \multimap \text{Smith's method}$$

$$X_1 = \hat{b}\hat{b}^* \qquad X_2 = \hat{b}\hat{b}^* + \hat{A}\hat{b}\hat{b}^*\hat{A}^*$$

$$X_3 = \hat{b}\hat{b}^* + \hat{A}\hat{b}\hat{b}^*\hat{A}^* + \hat{A}^2\hat{b}\hat{b}^*(\hat{A}^*)^2$$

$$X_k = \sum_{i=0}^{k-1} \hat{A}^i \hat{b}\hat{b}^* (\hat{A}^*)^i.$$

In terms of low-rank factors:

$$X_k = Z_k Z_k^*, \text{ with } Z_k = [\hat{b}, \hat{A}\hat{b}, \hat{A}^2\hat{b}, \dots \hat{A}^{(k-1)}\hat{b}] \in \mathbb{C}^{n\times k}$$

$$\begin{cases} v_0 = \hat{b} \\ v_{k+1} = \hat{A} v_k \end{cases} \quad k = 0,1,2,\dots \qquad Z_k = [v_0 \; v_1 \; \cdots \; v_{k-1}]$$

$$v_{k+1} = \underbrace{(A-\tau I)^{-1}}(A+\tau I) v_k \qquad v_0 = \sqrt{2\tau}\,\underbrace{(A-\tau I)^{-1}} b$$

We only need to solve sparse linear systems!

$$v_{k+1} = (A-\tau I)^{-1}(A-\tau I + 2\tau I) v_k = v_k + 2\tau (A-\tau I)^{-1} v_k.$$

$$Z_k Z_k^* = [\hat{b} \mid \hat{A}\hat{b} \mid \hat{A}^2\hat{b} \mid \cdots \mid \hat{A}^{k-1}\hat{b}] \begin{bmatrix} \hat{b}^* \\ \hline \hat{b}^*\hat{A}^* \\ \hline \vdots \\ \hline \hat{b}^*(\hat{A}^*)^{k-1} \end{bmatrix}$$

$$= \hat{b}\hat{b}^* + \hat{A}\hat{b}\hat{b}^*\hat{A}^* + \dots + \hat{A}^{k-1}\hat{b}\hat{b}^*(\hat{A}^*)^{k-1}.$$

Note that $\|\hat{A}^k\| \sim \rho(\hat{A})^k$, so the sum converges.

X_k converges to a fixed point $X_* = \hat{b}\hat{b}^* + \hat{A} X_* \hat{A}^*$, which must be the exact solution of the equation

$$X_* = \sum_{i=0}^{\infty} (\hat{A}^i)\, \hat{b}\hat{b}^* (\hat{A}^*)^i$$

$$X_* - Z_k Z_k^* = \sum_{i=k}^{\infty} \hat{A}^i \hat{b}\hat{b}^* (\hat{A}^*)^i = \hat{A}^k \left(\sum_{i=0}^{\infty} \hat{A}^i \hat{b}\hat{b}^* (\hat{A}^*)^i \right) (\hat{A}^*)^k$$

$$= \hat{A}^k X_* (\hat{A}^*)^k$$

$$\| X_* - Z_k Z_k^* \| \le \| \hat{A}^k \|^2 \, \| X_* \| \sim \rho(\hat{A})^{2k}$$ converges to 0, since $\rho(\hat{A}) < 1$

Convergence rate:

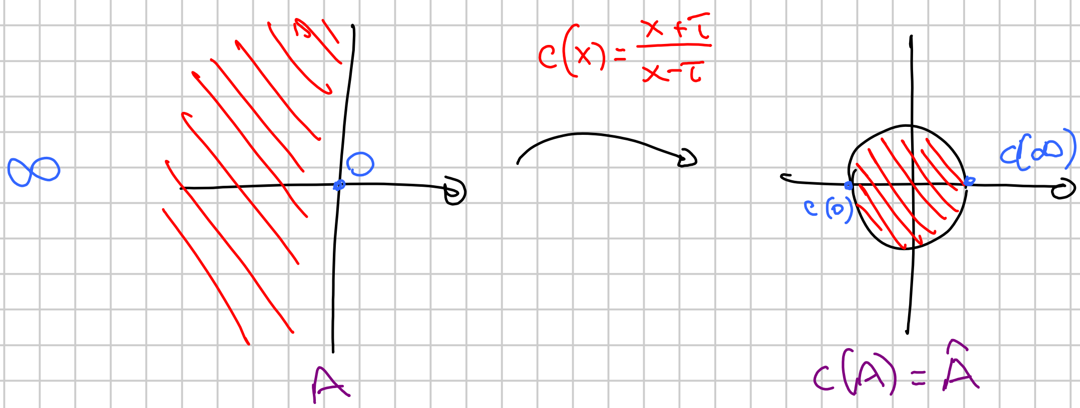
$$\max_{\lambda \in \Lambda(A)} \left| \frac{\lambda + \tau}{\lambda - \tau} \right|^2 < 1$$

If $\tau \gg |\lambda|$, $\dfrac{\lambda+\tau}{\lambda-\tau} \approx -1 \to$ slow convergence

If $\tau \ll |\lambda|$ $\dfrac{\lambda+\tau}{\lambda-\tau} \approx 1 \to$ slow convergence.

If A has all eigenvalues close, $\tau \approx \lambda$ is best

If A has both small and large eigenvalues, no luck.



Idea: change τ at every iteration

$A \qquad \hat{A}_k = (A-\tau_k I)^{-1}(A+\tau_k I) \qquad \hat{b}_k = \sqrt{2\tau_k}\,(A-\tau_k I)^{-1} b$

X solves $\qquad X = \hat{A}_k X \hat{A}_k^* + \hat{b}_k \hat{b}_k^*$ for each k.

$\tau_1, \tau_2, \tau_3, \dots$

$$X_k = \hat{A}_k X_{k-1} \hat{A}_k^* + \hat{b}_k \hat{b}_k^* \qquad X_0 = 0$$

$X_1 = \hat{b}_1 \hat{b}_1^*$

$X_2 = \hat{b}_2 \hat{b}_2^* + \hat{A}_2 \hat{b}_1 \hat{b}_1^* \hat{A}_2^*$

$X_3 = \hat{b}_3 \hat{b}_3^* + \hat{A}_3 b_2 b_2^* \hat{A}_3^* + \hat{A}_3 \hat{A}_2 \hat{b}_1 \hat{b}_1^* \hat{A}_2^* \hat{A}_3^*$

$\vdots$

$$Z_k = \begin{bmatrix} \hat{b}_k & \hat{A}_k \hat{b}_{k-1} & \cdots & \hat{A}_k \hat{A}_{k-1} \cdots \hat{A}_2 \hat{b}_1 \end{bmatrix}$$

$\boxed{\sqrt{2\tau_k}\,(A-\tau_k I)^{-1} b} \qquad \boxed{(A-\tau_k I)^{-1}(A+\tau_k I)(A-\tau_{k-1} I)^{-1} b \sqrt{2\tau_{k-1}}}$

v_0

$$v_1 = (A-\tau_{k-1} I)^{-1}(A+\tau_k I)(A-\tau_k I)^{-1} b \sqrt{2\tau_{k-1}}$$

$$= (A-\tau_{k-1} I)^{-1}(A+\tau_k I) v_0 \frac{\sqrt{2\tau_{k-1}}}{\sqrt{2\tau_k}}$$

ADI with multiple shifts:

$\alpha_0 = \tau_k \qquad \alpha_1 = \tau_{k-1} \qquad \alpha_2 = \tau_{k-2}, \dots$

$v_0 = \sqrt{2\alpha_0}\,(A-\alpha_0 I)^{-1} b$

$$v_j = \frac{\sqrt{2\alpha_j}}{\sqrt{2\alpha_{j-1}}}\left(v_{j-1} + (\alpha_{j-1}+\alpha_j)(A-\alpha_j I)^{-1} v_{j-1}\right)$$

Computing the residual:

$$\begin{cases} X_k = \hat{A}_k X_{k-1} \hat{A}_k^* + \hat{b}_k \hat{b}_k^* \\ X_* = \hat{A}_k X_* \hat{A}_k^* + \hat{b}_k \hat{b}_k^* \end{cases}$$

$$X_k - X_* = \hat{A}_k (X_{k-1} - X_*) A_k^*$$

$$X_k - X_* = \underbrace{\hat{A}_k \hat{A}_{k-1} \cdots \hat{A}_1}_{= r_k(A)} (X_0 - X_*) \underbrace{\hat{A}_1^* \hat{A}_2^* \cdots \hat{A}_k^*}_{= r_k(A)^*}$$

$$r_k(\lambda) = \frac{\lambda + \tau_k}{\lambda - \tau_k} \cdot \frac{\lambda + \tau_{k-1}}{\lambda - \tau_{k-1}} \cdot \cdots \cdot \frac{\lambda + \tau_1}{\lambda - \tau_1}$$

$$\|X_k - X_*\| \leq \|r_k(A)\|^2 \cdot \|X_0 - X_*\|$$

We want $\|r_k(A)\|$ as small as possible for convergence:

If A symmetric,

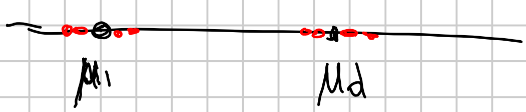
$$\|r_k(A)\| = \text{max eigenvalue} = \max_{\lambda \in \Lambda(A)} \left| \boxed{\frac{\lambda + \tau_k}{\lambda - \tau_k}} \cdot \boxed{\frac{\lambda + \tau_{k-1}}{\lambda - \tau_{k-1}}} \cdot \cdots \cdot \boxed{\frac{\lambda + \tau_1}{\lambda - \tau_1}} \right|.$$

If A has only a few distinct eigenvalues,

$\lambda_1, \ldots \lambda_d$,

we can choose $\tau_1 = -\lambda_1, \ldots \tau_d = -\lambda_d$, and after d steps we have exact convergence, $r_d(A) = 0$

If A has d clusters of eigenvalues centered in $\mu_1, \ldots \mu_d$,



we choose $\tau_i = -\mu_i$

We can formulate an optimization problem for each new shift:

Wachspress shifts

$$\alpha_k = \arg\min \max_{\lambda \in \Lambda(A)} \left|\frac{\lambda+\alpha_k}{\lambda-\alpha_k}\right| \cdot \left|\frac{\lambda+\alpha_{k-1}}{\lambda-\alpha_{k-1}}\right| \cdot \ldots \cdot \left|\frac{\lambda+\alpha_1}{\lambda-\alpha_1}\right|$$

To solve this, we would require the whole $\Lambda(A)$, and this is unfeasible

Workaround: compute approximations of the larger and smaller eigenvalues of A with Arnoldi, and then use them instead of $\Lambda(A)$ to solve the problem.

More theoretical solution: assume $\Lambda(A) \subseteq [a, b]$.

Then, we look for polynomials of degree k that achieve

$$\alpha(k) = \min_{p \in \mathcal{P}_k} \max_{\lambda \in [a,b]} \left|\frac{p(\lambda)}{p(-\lambda)}\right| \qquad \text{(Zolotarev problem)}$$

People know solutions that achieve the optimal decrease,

$\alpha(k) \sim r^k$ for a given $r < 1$, "logarithmic capacity" of $[a, b]$.

$$Z_k = \left[\hat{b}_k \quad \hat{A}_k \hat{b}_{k-1} \quad \cdots \quad \hat{A}_k \hat{A}_{k-1} \cdots \hat{A}_2 \hat{b}_1\right]$$

$$\hat{A}_k = (A - \tau_k I)^{-1}(A + \tau_k I)$$

$$\hat{b}_k = \sqrt{2\tau_k}\,(A - \tau_k I)^{-1} b$$

All columns of Z_k belong to

$$\operatorname{span}\left((A-\tau_1 I)^{-1} b,\ (A-\tau_2 I)^{-1} b,\ \dots\ (A-\tau_k I)^{-1} b \right)$$

$$= \operatorname{span}\left(r(A)b \text{ s.t. } r(z) \text{ is a rational function with denominator } (z-\tau_1 I)(z-\tau_2 I)\cdots(z-\tau_k I) \right)$$

= rational Krylov subspace

<u>Idea:</u> first compute a basis for the space V_k, then choose an approximation of the solution of the form

$$X_k = V_k Y_k V_k^*$$

$Y_k \in \mathbb{C}^{k\times k}$

Projected equation:

$$V_k^* A V_k Y_k V_k^* V_k + V_k^* V_k Y_k V_k^* A^* V_k + V_k^* b b^* V_k = 0$$

$$A_k Y_k + Y_k A_k^* + (V_k^* b)(b^* V_k) = 0.$$ A $k\times k$ equation

(rational Krylov algorithms)

$A_k = V_k^* A V_k$