GALOIS GROUPS AND FUNDAMENTAL GROUPS
Homework 1

1. Let k be a field with fixed separable closure k_s. Consider the set-valued functor F on the category $\mathcal{C}$ of finite étale k-algebras sending an algebra A to the set of k-algebra homomorphisms $A \to k_s$.

Determine the automorphism group of the functor F. [*Warning:* k_s is not an object of $\mathcal{C}$!]

2. Let $p : Y \to X$ be a cover of a connected and locally simply connected topological space X. For a point $x \in X$, we have defined two canonical left actions on the fibre $p^{-1}(x)$: one is the action by $\mathrm{Aut}(Y|X)$ and the other is that by $\pi_1(X, x)$. Verify that these two actions commute, i.e. that $\alpha(\phi(y)) = \phi(\alpha y)$ for $y \in \pi^{-1}(x)$, $\phi \in \mathrm{Aut}(Y|X)$ and $\alpha \in \pi_1(X, x)$.

3. Let X be a connected and locally simply connected topological space, and let $\widetilde{X}_x$ be the universal cover corresponding to a fixed base point $x \in X$. Consider a connected cover $p : Y \to X$ and a base point $y \in p^{-1}(x)$.

a) Show that the homomorphism $\pi_1(Y, y) \to \pi_1(X, x)$ induced by p is injective.

b) Viewing $\pi_1(Y, y)^{op}$ as a subgroup of $\mathrm{Aut}(\widetilde{X}_x|X)$ via the isomorphism $\mathrm{Aut}(\widetilde{X}_x|X) \cong \pi_1(X, x)^{op}$, establish an isomorphism $\widetilde{X}_x/\pi_1(Y, y)^{op} \cong Y$.