GALOIS GROUPS AND FUNDAMENTAL GROUPS
Homework 2

1. Let $Y \to X$ be a connected cover of topological spaces, and let $G := \mathrm{Aut}(Y|X)$. Show that $Y \to X$ is a Galois cover with group G if and only if the map $(y, g) \mapsto (y, g(y))$ induces an isomorphism of covers between the trivial cover $Y \times G \to Y$ and the fibre product $Y \times_X Y \to Y$ (here G carries the discrete topology).

2. Let $Y \to X$ be a holomorphic map of compact Riemann surfaces with X connected, restricting to a cover $Y' \to X'$ outside the branch points.

a) Show that the étale $\mathcal{M}(X)$-algebra $\mathcal{M}(Y)$ is isomorphic to a finite direct sum of copies of $\mathcal{M}(X)$ if and only if the cover $Y' \to X'$ is trivial.

b) Using Exercise 1 give another proof of the fact that in the anti-equivalence between compact Riemann surfaces mapping holomorphically onto X and finite étale $\mathcal{M}(X)$-algebras Galois branched covers of X correspond to finite Galois field extensions of $\mathcal{M}(X)$.

3. *a*) Let k be an algebraically closed field of characteristic not 2, $f \in k[x_1]$ a nonconstant polynomial such that $x_2^2 - f$ is irreducible in $k[x_1, x_2]$. Consider $Y = V(x_2^2 - f) \subset \mathbf{A}_k^2$ and $\phi : Y \to \mathbf{A}_k^1$ the morphism given by $(x_1, x_2) \mapsto x_1$. Show that ϕ is a finite morphism that is étale over the point of $\mathbf{A}_k^1$ corresponding to $a \in k$ if and only if $f(a) \neq 0$.

b) Does the same conclusion hold when k is of characteristic 2?