GALOIS GROUPS AND FUNDAMENTAL GROUPS
Homework 3

1. Let k be a field of characteristic 0 containing the n-th roots of unity.

a) Show that for every $a \in k$ there exists a Galois field extension $K_n|k(t)$ with group $\mathbf{Z}/n\mathbf{Z}$ such that the normalisation X of $\mathbf{A}^1_k$ in K_n is étale above the complement of the point of $\mathbf{A}^1_k$ defined by a, and moreover there is only one point of X above a, with inertia subgroup equal to the whole of $\mathbf{Z}/n\mathbf{Z}$.

b) Show that if $a \neq a' \in k$, then $K_a \cap K_{a'} = k(t)$.

2. Let k again be a field of characteristic 0 containing the n-th roots of unity, and let A be a finite abelian group with $nA = 0$.

Show that there exists a finite Galois extension $L|k(t)$ with group A such that k is algebraically closed in L.

3. Let k be an algebraically closed field of characteristic $p > 0$, and consider the rational function field $k(t)$. Choose $f \in k[t]$ so that the polynomial $x^p - x - f \in k(t)[x]$ is irreducible over $k(t)$.

Show that the normalization of $\mathbf{A}^1_k$ in the finite extension of $k(t)$ defined by $x^p - x - f$ is an étale Galois cover of $\mathbf{A}^1_k$ whose Galois group is $\mathbf{Z}/p\mathbf{Z}$. (It is called an *Artin–Schreier cover* of $\mathbf{A}^1_k$.)

4. Let k again be an algebraically closed field of characteristic $p > 0$, and let $\pi_1^{\mathrm{ab}}(\mathbf{A}^1_k)$ be the maximal abelian quotient of the profinite group $\pi_1(\mathbf{A}^1_k)$, i.e. its quotient modulo the (closed) commutator subgroup.

Using the previous exercise, show that $\pi_1^{\mathrm{ab}}(\mathbf{A}^1_k)/p\pi_1^{\mathrm{ab}}(\mathbf{A}^1_k)$ is an infinite-dimensional $\mathbf{F}_p$-vector space.

[Bonus question: can you determine the cardinality of $\pi_1^{\mathrm{ab}}(\mathbf{A}^1_k)/p\pi_1^{\mathrm{ab}}(\mathbf{A}^1_k)$?]