GALOIS GROUPS AND FUNDAMENTAL GROUPS
Oral exam questions, 2025/26

1. Show that the Galois group of an infinite Galois extension is profinite, and in particular a compact Hausdorff topological group. State the Galois correspondence for infinite Galois extensions.

2. State and prove Grothendieck's formulation of Galois theory.

3. State the correspondence between intermediate covers of a Galois topological cover and subgroups of the Galois group. Give the proof that Galois intermediate covers correspond to normal subgroups.

4. Prove that the fibre functor on the category of covers of a connected and locally simply connected topological space is representable by a cover whose automorphism group is the opposite group of the fundamental group. (The proof that it is a connected Galois cover is not needed.)

5. Assuming the statements of the previous two questions, explain the equivalence of categories between covers of a connected and locally simply connected topological space X and sets equipped with the monodromy action of the fundamental group of X at a base point.

6. Explain the correspondence between complex local systems on a topological space and finite-dimensional complex representations of the fundamental group.

7. Explain the correspondence between finite branched covers of a compact Riemann surface and finite étale algebras over its field of meromorphic functions. (Proofs of intermediate lemmas not needed.)

8. Show how to realize the profinite completion of the fundamental group of a compact Riemann surface minus finitely many points as the Galois group of a certain infinite Galois extension of its field of meromorphic functions.

9. Explain how to put the structure of a Riemann surface on the points of an integral normal curve over $\mathbf{C}$. Show that in the case of a finite morphism of such curves the algebraic ramification index at a point equals the ramification index of the corresponding branched cover.

10. Construct the algebraic fundamental group of an open subset of an integral proper normal curve and show that when the base field is $\mathbf{C}$ it equals the profinite completion of the topological fundamental group of the associated Riemann surface. (Proofs of intermediate lemmas not needed.)

11. Explain the idea of the rigidity method for realizing certain noncommutative finite groups as Galois groups of a regular Galois extension of $\mathbf{Q}(t)$.

12. State Belyi's theorem on finite étale covers of the projective line minus three points over $\mathbf{Q}$ and show how it implies the faithfulness of the corresponding outer Galois representation on the geometric fundamental group.