Matrix pencils

Definition: Matrix pencil

$A + xB$, with $A, B \in \mathbb{C}^{m \times n}$, x indeterminate.

A pencil is called regular if $n = m$ and $\det(A + xB)$ does not vanish identically, i.e., if there is $\lambda \in \mathbb{C}$ for which it is square invertible.

An eigenvalue λ is a value for which $\det(A + \lambda B) = 0$.
Eigenvector, Jordan chains...

If $\det(A + xB)$ has degree less than n, the 'missing' eigenvalues are said to be "at infinity".

Example

$$\overbrace{\begin{bmatrix} x+1 & x \\ x & x+1 \end{bmatrix}}^{\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix} x + \begin{bmatrix}1 & 0 \\ 0 & 1\end{bmatrix}} \quad \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}$$

$\det\begin{bmatrix} 1 & x \\ 1 & x \end{bmatrix} = x - x = 0 \rightsquigarrow$ singolare

$\det\begin{bmatrix} 1+x & x \\ x & 1+\cancel{x} \end{bmatrix} = 2x+1 \rightsquigarrow$ regolare

se $x = -\frac{1}{2}$, viene $\begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \rightsquigarrow$ non invertibile

$\rightsquigarrow \lambda = -\frac{1}{2}$ è un autovalore di $\begin{bmatrix} 1+x & x \\ x & 1+x \end{bmatrix}$

$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ autovettore corrispondente

(autoval/vett. di una matrice $M \Leftrightarrow$ di $M - \cancel{x}I$)

$\det(A+Bx)$ has degree at most n

If B is singular, lower degree

<u>Def</u>: A pencil has $n - \deg \det(A+Bx)$ eigenvalues at infinity

[remark: $A+Bx$ has an eigenvalue at ∞
iff $Ax+B$ has an eigenvalue at 0]

$\begin{bmatrix} x+1 & x \\ x & x+1 \end{bmatrix}$: eigvl at $-\frac{1}{2}, \infty$

$\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}$: eigvl at ∞ with multiplicity 2

Eigenvalues of singular pencils

Can be defined via 'unusual rank drop'. For instance:

$$A + xB = \begin{bmatrix} 2 & x & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ x & x & 0 \end{bmatrix}$$

$2-x$

0

$\left|\begin{matrix} 2 & x \\ x & x \end{matrix}\right| = 2x - x^2$

has typical rank 2. More formally, $\mathrm{rank}_{\mathbb{C}(x)}(A + xB) = 2$.
But $A + 2B$ has rank 1.

Remark: for almost all $\lambda \in \mathbb{C}$ (apart from a finite set), $\mathrm{rank}_{\mathbb{C}}(A + \lambda B) = \mathrm{rank}_{\mathbb{C}(x)}(A + xB)$.

Canonical form

Equivalence relation $\sim$: for each two square $P \in \mathbb{C}^{m\times m}$, $Q \in \mathbb{C}^{n\times n}$ square invertible, $A + xB$ and $P(A + xB)Q$ are said to be equivalent.

$PAQ + x \cdot PBQ$

Equivalent $\implies$ same eigenvalues, singularity...

If B is square nonsingular, there is little new in this theory: $A + xB \sim J - xI$, where J is the Jordan canonical form of $-B^{-1}A$ (or $-AB^{-1}$).

Computing eigenvalues of $A + xB \iff$ computing eigenvalues of $-B^{-1}A$

$$\det P(A+xB)Q = \det P \det(A+xB) \det Q$$

RMK:

$M - xI \sim N - xI \iff M$ e N simili,

$M = SNS^{-1}$

Se B invertibile:

$$A+xB \sim -B^{-1}(A+xB) = -B^{-1}A - xI \sim$$

$$\sim S(-B^{-1}A)S^{-1} - xI = J - xI$$

Jordan form of $-B^{-1}A$

Theorem (Weierstrass canonical form)

For a regular matrix pencil $A + xB \in \mathbb{C}[x]^{n\times n}$, there are nonsingular $P, Q \in \mathbb{C}^{n\times n}$ such that $P(A+xB)Q$ is the direct sum (`blkdiag`) of blocks of the forms

$$J_\lambda(x) = \underbrace{\begin{bmatrix} \lambda & 1 & & \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{bmatrix} - xI}_{\text{Jordan block at } \lambda}, \qquad \underbrace{J_\infty(x) = I - x\begin{bmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix}}_{\text{Jordan block at } \infty}.$$

$$\begin{bmatrix} \lambda - x & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda - x \end{bmatrix} \qquad \begin{pmatrix} 1 & -x & & 0 \\ & \ddots & \ddots & \\ & & \ddots & -x \\ 0 & & & 1 \end{pmatrix}$$

Proof (sketch):

- Take c such that $A + cB$ is invertible;
- $A + xB \sim I + (x - c)(A + cB)^{-1}B$;
- $A + xB \sim I + (x - c)\,\mathrm{blkdiag}(J_1, \dots, J_s)$,
- Consider separately each $I + (x - c)J_i = I + (x - c)(\lambda I + N)$.
- If $\lambda = 0$, block $\sim I - xM$, where $M = \mathrm{toeplitztriu}(0, 1, \dots)$.
- If $\lambda \neq 0$, block $\sim M - xI$, where $M = \mathrm{toeplitztriu}(\frac{c\lambda-1}{\lambda}, \frac{1}{\lambda^2}, \dots)$.

Prendo $c \in \mathbb{C}$ tale che $A+cB$ sia invertibile

$$A+xB = (A+cB)+(x-c)B \sim$$

$$I+(x-c)\underbrace{(A+cB)^{-1}B}\sim \left[\text{Jordan form of } (A+cB)^{-1}B\right]$$

$$\sim I+(x-c)\,\mathrm{diag}(J_1, J_2, \dots J_s)$$

$$= \begin{bmatrix} I+(x-c)J_1 & & & 0 \\ & I+(x-c)J_2 & & \\ & & \ddots & \\ 0 & & & I+(x-c)J_s \end{bmatrix}$$

Lavoro sui singoli blocchi:

$$J_i = \lambda I + N \qquad N = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix}$$

$$I + (x-c) J_i = \underline{(1-c\lambda) I - cN + x(\lambda I + N)} \quad \boxed{*}$$

CASO 1 $\lambda = 0$

$$\boxed{*} = (I - cN) + xN \sim I + x\underset{\Vert}{(I - cN)^{-1}} N$$

$$(I + cN + c^2N^2 + c^3N^3 + \dots) N$$

$$= I + x \overbrace{\begin{bmatrix} 0 & 1 & c & c^2 & \cdots \\ & \ddots & \ddots & \ddots & c \\ & & \ddots & \ddots & c \\ & 0 & & \ddots & 1 \\ & & & & 0 \end{bmatrix}}^{K_i}$$

one eigenvalue at 0

$$rK\left(\begin{bmatrix} 0 & 1 & c & \cdots & c \\ & \ddots & \ddots & & \vdots \\ & & \ddots & & 1 \\ & 0 & & & 0 \end{bmatrix} - 0 \cdot I \right) = K_i - 1$$

Since $\dim\operatorname{Ker}\begin{bmatrix} 0 & 1 & c & \cdots & c \\ & \ddots & \ddots & \ddots & \vdots \\ & & \ddots & \ddots & c \\ & 0 & & \ddots & 1 \\ & & & & 0 \end{bmatrix} = 1$, it has a single Jordan block,

$$I - x\begin{bmatrix} 0 & 1 & c & \cdots & c \\ & \ddots & \ddots & \ddots & \vdots \\ & & \ddots & \ddots & c \\ & & & \ddots & 1 \\ & & & & 0 \end{bmatrix} \sim I - x\begin{bmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{bmatrix}.$$

② $\lambda \neq 0$

$$\circledast \sim \big((1-c\lambda)I - cN\big)\underbrace{(\lambda I + N)}^{-1} - xI$$

(ragionamento analogo)

One can define 'Jordan chains' (at λ, at ∞. . .)

Generalized Schur factorization

Compare with generalized Schur (QZ) factorization:

Theorem

For any pair of square $A, B \in \mathbb{C}^{m\times m}$, one can find orthogonal Q, Z such that $QAZ = T_A, QBZ = T_B$ are upper triangular (at the same time).

Eigenvalues $= \frac{(T_A)_{ii}}{(T_B)_{ii}}$ (incl. ∞).

Theorem (Kronecker canonical form)

For a ~~regular~~ matrix pencil $A + xB \in \mathbb{C}[x]^{m\times n}$, there are nonsingular $P \in \mathbb{C}^{m\times m}, Q \in \mathbb{C}^{n\times n}$ such that $P(A + xB)Q$ is the direct sum (`blkdiag`) of blocks of the form $J_\lambda(x), J_\infty(x)$, and

$$\begin{bmatrix} 1 & x & & & \\ & 1 & x & & \\ & & \ddots & \ddots & \\ & & & 1 & x \end{bmatrix} \in \mathbb{C}[x]^{k\times(k+1)}, \quad \begin{bmatrix} 1 & & & \\ x & 1 & & \\ & x & \ddots & \\ & & \ddots & 1 \\ & & & x \end{bmatrix} \in \mathbb{C}[x]^{(k+1)\times k},$$

(This includes 1×0 and 0×1 empty blocks).

Examples

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & x \\ 1 & 0 & 0 \\ x & 0 & 0 \end{bmatrix} \ldots$$

Proof (sketch): [Gantmacher book '59]

- Suppose $(A + xB)v(x) = 0$ for some $v \in \mathbb{C}(x)^n$
- We may assume $v = v_0 + v_1 x + \cdots + v_d x^d \in \mathbb{C}[x]^n$, clearing denominators.
- Remark: singularity of $(d+1) \times d$ $\begin{bmatrix} A & & & \\ B & A & & \\ & \ddots & \ddots & \\ & & B & A \\ & & & B \end{bmatrix}$.
- Assume d minimal.
- We wish to show that the v_i are linearly independent. Suppose they are not so; then one can choose $\alpha(x) = \alpha_0 + \alpha_1 x + \cdots + \alpha_e x^e$ (of minimal degree $e \leq d$) such that $w(x) = \alpha(x)v(x)$ has a zero coefficient w_e. But then $Aw_0 = 0$, $Aw_1 + Bw_0 = 0$, $\ldots$, $Bw_{e-1} = 0$, which contradicts minimality of d.

(cont.)

- Take a basis that starts with the v_i; this block-triangularizes the pencil: $\begin{bmatrix} K(x) & L(x) \\ 0 & M(x) \end{bmatrix}$, where $K(x)$ is a Kronecker block.
- Moreover, by minimality of d, $M(x)$ is such that $d \times (d-1)$
$$\begin{bmatrix} M_0 & & & \\ M_1 & M_0 & & \\ & \ddots & \ddots & \\ & & M_1 & M_0 \\ & & & M_1 \end{bmatrix}$$
is nonsingular.
- Using this nonsingularity, one can prove that the system of Sylvester-like equations
$$\begin{bmatrix} I & E \\ 0 & I \end{bmatrix} \begin{bmatrix} K(x) & L(x) \\ 0 & M(x) \end{bmatrix} \begin{bmatrix} I & F \\ 0 & I \end{bmatrix} = \begin{bmatrix} K(x) & {\color{red}0} \\ 0 & M(x) \end{bmatrix}$$
is solvable (some work needed — details not given in the course).

Kernel in $\mathbb{C}(x)$

The $(k \times (k+1))$ Kronecker blocks have kernel
$\begin{bmatrix}(-1)^k x^k & (-1)^{k-1} x^{k-1} & \dots & -x & 1\end{bmatrix}^T$.
The other blocks have full column rank in $\mathbb{C}(x)$.
(Remark: the kernel of `blkdiag`(C,D) can be obtained by the kernels of C, D.)