Matrix polynomials

A matrix polynomial is $A(x) = A_0 + A_1 x + A_2 x^2 + \cdots + A_d x^d$.

We assume for now $A_i \in \mathbb{C}^{m \times m}$.

d "not exactly" degree: we admit zero leading coefficients.

Eigenvalues/vectors are pairs such that $A(\lambda)v = 0$.

If the polynomial is regular ($\det A(x)$ is not identically zero), then there is at most dm of them. They can be at ∞, like for pencils.

Reversal and infinite eigenvalues

Reversal of a matrix polynomial: same coefficients but in the opposite order:

$$\operatorname{Rev}(A_0 + A_1 x + A_2 x^2 + A_3 x^3) = A_3 + A_2 x + A_1 x^2 + A_0 x^3.$$

Lemma

Let $A(x)$ be a regular matrix polynomial. The eigenvalues of $\operatorname{Rev} A(x)$ are $\frac{1}{\lambda_i}$, where λ_i are the eigenvalues of $A(x)$. This includes also eigenvalues at ∞, with the convention that $\frac{1}{0} = \infty$ and $\frac{1}{\infty} = 0$.

Proof: direct verification for $\lambda \notin \{0, \infty\}$. Homogenize $\det A(x)$ to count eigvls at 0 and ∞.

The companion linearization

Theorem

Let $A(x) = A_0 + A_1 x + A_2 x^2 + \cdots + A_d x^d$ be a matrix polynomial. Then, $\det A(x) = \det C(x)$, where $C(x)$ is the pencil ("Frobenius companion form")

$$C(x) = \begin{bmatrix} A_d x + A_{d-1} & A_{d-2} & \dots & A_1 & A_0 \\ I_m & -xI_m & & & \\ & I_m & -xI_m & & \\ & & \ddots & \ddots & \\ & & & I_m & -xI_m \end{bmatrix} \in \mathbb{C}[x]^{dm \times dm}.$$

We prove something stronger: there are $E(x), F(x) \in \mathbb{C}(x)^{dm \times dm}$ with determinant 1 s.t. $E(x)C(x)F(x) = \operatorname{blkdiag}(A(x), I_{(d-1)m})$.

Proof (sketch): make linear combinations of columns to eliminate the blocks $-xI$.

Other linearizations

Other pencils with the same property $E(x)C(x)F(x) = \mathrm{blkdiag}(A(x), I_{(d-1)m})$ can be constructed — they are called linearizations.

Some final projects available on methods to construct them.

Eigenvector recovery

Theorem

$v \neq 0$ is an eigenvector of $A(x)$ (with eigvl $\lambda \neq \infty$) iff

$$w(\lambda, v) = \begin{bmatrix} \lambda^{d-1} v \\ \lambda^{d-2} v \\ \vdots \\ v \end{bmatrix}$$

is an eigenvector of $C(x)$.

Proof: direct verification. Start calling v the last block of $w \ldots$.

Eigenvectors are not independent

Small surprise: the eigenvectors of $A(x)$ are (usually) not linearly independent.
(How could they be? There are too many of them...)
However, $w(\lambda_i, v_i)$ are linearly independent.

Application

What do we use linearization / eigenvalues of matrix polynomials for?
Linear differential equations: (assume $\det(A_2) \neq 0$)

$$A_2\ddot{x} + A_1\dot{x} + A_0 x = 0, \quad x : [t_0, t_f] \to \mathbb{R}^n. \qquad \text{(ode)}$$

Special solutions: $e^{\lambda t} v$, where (λ, v) eigenpair of the matrix polynomial.

General solution: via linearization of $A_2 x^2 + A_1 x + A_0$ (matrix exponential of $-C_1^{-1} C_0$).

(Stable solutions: invariant subspace formed by eigenvalues with $\operatorname{Re} \lambda < 0$.)

What about polynomials with eigenvalues at ∞ / singularities?
More involved — differential-algebraic equations. Example
$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$.

Jordan chains [Gohberg-Lancaster-Rodman book, Sec. 1.4]

One can define Jordan chains of a matrix polynomial using derivatives:

$$
\begin{aligned}
&A(\lambda)v_0 = 0,\\
&A'(\lambda)v_0 + A(\lambda)v_1 = 0,\\
&\frac{1}{2}A''(\lambda)v_0 + A'(\lambda)v_1 + A(\lambda)v_2 = 0\\
&\vdots
\end{aligned}
$$

With this definition, $v_0 e^{\lambda t}$, $(v_0 t + v_1)e^{\lambda t}$, $(v_0 t^2 + v_1 t + v_2)e^{\lambda t}$, ... are special solutions of (ode).

How to define Jordan chains at ∞? As Jordan chains at zero of $\operatorname{Rev} A(x)$.

The problem with linearizations and $\lambda = \infty$

The relation $C(x) \sim D(x)$ iff $C(x) = E(x)D(x)F(x)$ preserves the sizes of Jordan chains at $\lambda \in \mathbb{C}$ (why?), but not of those at infinity. This can be seen already for degree 1: the pencils

$$D(x) = I + 0x = \begin{bmatrix} 1 & \\ & 1 \end{bmatrix}$$

and

$$C(x) = \begin{bmatrix} 1 & x \\ & 1 \end{bmatrix} = D(x)F(x) \quad \text{with } F(x) = \begin{bmatrix} 1 & x \\ & 1 \end{bmatrix}$$

have different Jordan structures at ∞.

Strong linearizations

A linearization is said to be strong if $\operatorname{Rev} L(x)$ is also a linearization of $\operatorname{Rev} A(x)$.

Result

The companion pencil $C(x)$ is a strong linearization.

Proof: again linear combinations of columns to 'fold up' the polynomial $\operatorname{Rev} A(x)$.

Smith form [Gohberg–Lancaster–Rodman book, appendix S1]

Quick review of other invariants for matrix polynomials...

Smith normal form

There are matrices $E(x)$, $F(x)$ with determinant 1 such that $E(x)A(x)F(x) = \mathrm{diag}(d_1(x), d_2(x), \dots, d_r(x), 0, 0, \dots, 0)$, and $d_i(x) \mid d_{i+1}(x)$ for all i.

Actually an algebra result — holds in every PID. (Proof idea: a sort of back-and-forth Gaussian elimination like the one used to compute inverses. Instead of division, use Bézout identities).
The d_is are uniquely defined (GCDs of all $i \times i$ minors).
Reveals:

- Rank over $\mathbb{C}(x)$;
- Eigenvalues (roots of $d_r(x)$);
- Sizes of Jordan chains (depend on how many $d_i(\lambda)$ vanish).

Minimal indices [Forney, '72]

Quick review of other invariants for matrix polynomials...
The generalization of sizes of singular Kronecker blocks are minimal indices.

Like in the degree-1 case, $\ker_{\mathbb{C}(x)} A(x)$ and $\ker_{\mathbb{C}(x)} A^T(x)$ admit polynomial bases with degrees "as small as possible" — these degrees are called minimal indices.

Linearizations do not always preserve minimal indices — but sometimes they change them predictably. For instance, for $C(x)$, right minimal indices are preserved, left minimal indices are increased by $d - 1$.