Matrix pencils

Definition: Matrix pencil

$A + xB$, with $A, B \in \mathbb{C}^{m \times n}$, x indeterminate.

A pencil is called regular if $n = m$ and $\det(A + xB)$ does not vanish identically, i.e., if there is $\lambda \in \mathbb{C}$ for which it is square invertible.

An eigenvalue λ is a value for which $\det(A + \lambda B) = 0$.
Eigenvector, Jordan chains. . .

If $\det(A + xB)$ has degree less than n, the 'missing' eigenvalues are said to be "at infinity".

Example

$$\begin{bmatrix} x+1 & x \\ x & x+1 \end{bmatrix} \quad \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}$$

Eigenvalues of singular pencils

Can be defined via 'unusual rank drop'. For instance:

$$A + xB = \begin{bmatrix} 2 & x & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ x & x & 0 \end{bmatrix}$$

has typical rank 2. More formally, $\operatorname{rank}_{\mathbb{C}(x)}(A + xB) = 2$.
But $A + 2B$ has rank 1.

Canonical form

Equivalence relation $\sim$: for each two square $P \in \mathbb{C}^{m\times m}, Q \in \mathbb{C}^{n\times n}$ square invertible, $A + xB$ and $P(A + xB)Q$ are said to be equivalent.

Equivalent $\implies$ same eigenvalues, singularity...

If B is square nonsingular, there is little new in this theory: $A + xB \sim J - xI$, where J is the Jordan canonical form of $-B^{-1}A$ (or $-AB^{-1}$).
Computing eigenvalues of $A + xB \iff$ computing eigenvalues of $-B^{-1}A$

Theorem (Weierstrass canonical form)

For a regular matrix pencil $A + xB \in \mathbb{C}[x]^{n\times n}$, there are nonsingular $P, Q \in \mathbb{C}^{n\times n}$ such that $P(A+xB)Q$ is the direct sum (`blkdiag`) of blocks of the forms

$$J_\lambda(x) = \begin{bmatrix} \lambda & 1 & & \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{bmatrix} - xI, \quad J_\infty(x) = I - x \begin{bmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix}.$$

Proof (sketch):

- Take c such that $A + cB$ is invertible;
- $A + xB \sim I + (x - c)(A + cB)^{-1}B$;
- $A + xB \sim I + (x - c)\,\mathrm{blkdiag}(J_1, \dots, J_s)$,
- Consider separately each $I + (x - c)J_i = I + (x - c)(\lambda I + N)$.
- If $\lambda = 0$, block $\sim I - xM$, where $M = \mathrm{toeplitztriu}(0, 1, \dots)$.
- If $\lambda \neq 0$, block $\sim M - xI$, where $M = \mathrm{toeplitztriu}(\frac{c\lambda - 1}{\lambda}, \frac{1}{\lambda^2}, \dots)$.

One can define Jordan chains: $P(A+xB)Q = \lambda I + N - xI$

at λ: $-Av_0 = \lambda B v_0$, $-Av_1 = \lambda B v_1 + B v_0, \dots$

at ∞: $-Bv_0 = 0$, $-Bv_1 = Av_0, \dots$

$A + xB$ (λ finito) Su un blocco:

$PAQ = \lambda I + N$ $\quad PBQ = -I$

$$N = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & & 1 \\ & & & 0 \end{bmatrix}$$

$$\underline{AQ} = P^{-1}(\lambda I + N) = \underline{-BQ(\lambda I + N)},$$

$$A\begin{bmatrix} v_0 & v_1 & \cdots & v_k \end{bmatrix} = -B\begin{bmatrix} v_0 & v_1 & \cdots & v_k \end{bmatrix}\begin{bmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & & 1 \\ 0 & & & \lambda \end{bmatrix}$$

$Av_0 = -\lambda B v_0$ $\leftarrow$ 1ª colonna

$Av_1 = -Bv_0 - \lambda B v_1$

$Av_2 = -Bv_1 - \lambda B v_2 \quad \dots$

Generalized Schur factorization

Compare with generalized Schur (QZ) factorization:

Theorem

For any pair of square $A, B \in \mathbb{C}^{m\times m}$, one can find orthogonal Q, Z such that $QAZ = T_A$, $QBZ = T_B$ are upper triangular (at the same time).

Eigenvalues $= \frac{(T_A)_{ii}}{(T_B)_{ii}}$ (incl. ∞).

$$Q(A+Bx)Z = \begin{bmatrix} \ddots & \\ 0 & \ddots \end{bmatrix} + \begin{bmatrix} \ddots & \\ 0 & \ddots \end{bmatrix} x = \begin{bmatrix} T_{A11}+xT_{B11} & & * \\ & T_{A22}+xT_{B22} & \\ 0 & & \ddots \\ & & & T_{Amm}+xT_{Bmm} \end{bmatrix}$$

Diventa singolare se sostituisco $x = -\frac{T_{Aii}}{T_{Bii}}$

se $T_{Bii}=0$, $T_{Aii}\neq 0$ $\rightsquigarrow$ autoval. all'infinito

se $T_{Bii}=T_{Aii}=0$ $\rightsquigarrow$ pencil singolare

Theorem (Kronecker canonical form)

For a ~~regular~~ matrix pencil $A + xB \in \mathbb{C}[x]^{m\times n}$, there are nonsingular $P \in \mathbb{C}^{m\times m}$, $Q \in \mathbb{C}^{n\times n}$ such that $P(A+xB)Q$ is the direct sum (`blkdiag`) of blocks of the form $J_\lambda(x)$, $J_\infty(x)$, and

$L_{k\times(k+1)}(x)$ ''

$$\begin{bmatrix} 1 & x & & & \\ & 1 & x & & \\ & & \ddots & \ddots & \\ & & & 1 & x \end{bmatrix} \in \mathbb{C}[x]^{k\times(k+1)}, \qquad \begin{bmatrix} 1 & & & \\ x & 1 & & \\ & x & \ddots & \\ & & \ddots & 1 \\ & & & x \end{bmatrix} \in \mathbb{C}[x]^{(k+1)\times k},$$

$(k, k+1)$

$L_{k+1\times k}(x)$

(This includes 1×0 and 0×1 empty blocks).

$\mathbb{R}^5 \leadsto \{0\}$

$-$ $\begin{bmatrix} 1 & x \end{bmatrix}$ $\begin{bmatrix} 1 & x & \\ & 1 & x \end{bmatrix}$

0×1

0×5

Examples

$$\overset{①}{\begin{bmatrix}0&0\\0&0\end{bmatrix}}, \quad \overset{②}{\begin{bmatrix}0&2\\0&1\end{bmatrix}}, \quad \overset{③}{\begin{bmatrix}1&x\\1&x\end{bmatrix}}, \quad \overset{④}{\begin{bmatrix}0&1&x\\1&0&0\\x&0&0\end{bmatrix}} \quad \dots \quad \overset{⑤}{\begin{bmatrix}*&*\\*&*\\*&*\end{bmatrix}}$$

① $= \operatorname{blkdiag}(0\times 1,\ 0\times 1,\ 1\times 0,\ 1\times 0)$

⑤ (deve avere un $L_{k+1,k}$)

② non in forma can.

$$\underset{P}{\begin{bmatrix}1&-2\\0&1\end{bmatrix}}\underset{A+0\cdot B}{\begin{bmatrix}0&2\\0&1\end{bmatrix}}\underset{Q}{I} = \begin{bmatrix}0&0\\0&1\end{bmatrix} = \left[\begin{array}{c|c}0&0\\ \hline 0&\boxed{1}\end{array}\right]$$

$\boxed{1}$: $J_\infty(x)$ blocco di Jordan all'infinito

Forma canonica di ②:

$$\operatorname{blkdiag}(L_{0\times 1},\ L_{1\times 0},\ J_\infty)$$

③ $$\underbrace{\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} 1 & x \\ 1 & x \end{bmatrix}}_{(Ax+B)} \underbrace{I}_{Q} = \begin{bmatrix} \boxed{1 \quad x} \\ 0 \quad 0 \end{bmatrix} = \mathrm{blkdiag}(L_{1\times 2}, L_{1\times 0})$$

④ $$\underbrace{\begin{bmatrix} 0 & 1 & x \\ 1 & 0 & 0 \\ x & 0 & 0 \end{bmatrix}}\, Q = \begin{bmatrix} 1 & x & 0 \\ 0 & 0 & 1 \\ 0 & 0 & x \end{bmatrix} = \mathrm{blkdiag}(L_{1\times 2}, L_{2\times 1})$$

(Falsa credenza:
se $A+xB$ singolare, allora $\exists\, v \neq 0 : Av = Bv = 0$
oppure $\exists\, w$ t.c. $w^T A = w^T B = 0$.)

(Controesempio: ④)

Proof (sketch): [Gantmacher book '59]

- Suppose $(A + xB)v(x) = 0$ for some $v \in \mathbb{C}(x)^n$
- We may assume $v = v_0 + v_1 x + \cdots + v_d x^d \in \mathbb{C}[x]^n$, clearing denominators.
- Remark: singularity of $(d+1) \times d$ $\begin{bmatrix} A & & & \\ B & A & & \\ & \ddots & \ddots & \\ & & B & A \\ & & & B \end{bmatrix}$.
- Assume d minimal.
- We wish to show that the v_i are linearly independent. Suppose they are not so; then one can choose $\alpha(x) = \alpha_0 + \alpha_1 x + \cdots + \alpha_e x^e$ (of minimal degree $e \leq d$) such that $w(x) = \alpha(x)v(x)$ has a zero coefficient w_e. But then $Aw_0 = 0$, $Aw_1 + Bw_0 = 0$, $\ldots$, $Bw_{e-1} = 0$, which contradicts minimality of d.

(cont.)

se $A+Bx$ singolare, allora $\exists\ v(x) \neq 0$ t.c. $(A+Bx)v(x) \equiv 0$

oppure $w^T(x) \neq 0$ t.c. $w^T(x)(A+Bx) \equiv 0$

Supponiamo che $\exists\ v(x) \in \mathbb{C}(x)^n$ t.c.

$$(A+Bx) \cdot v(x) \equiv 0$$

Posso prendere $v(x) \in \mathbb{C}[x]^n$ (levo denominat.)

<u>Oss</u>: se $v(x) = v_0 + v_1 x + \ldots + v_d x^d$, allora

$$\begin{array}{cc} (d+2)m & \begin{bmatrix} A & & & \\ B & A & & \\ & B & A & \\ & & \ddots & A \\ & & & B \end{bmatrix} \\ & (d+1)n \end{array} \begin{bmatrix} v_0 \\ v_1 \\ \vdots \\ v_d \end{bmatrix} = 0 \qquad \begin{array}{l} A v_0 = 0 \\ \\ \\ B v_d = 0 \end{array}$$

[**Lemma:** $v_0, \dots v_d$ sono lin. indipendenti (se prendo d minimo)]

Prendiamo una base che comincia con $[v_d, v_{d-1} \dots v_0]$ e una base che comincia con $[Av_d, Av_{d-1}, \dots Av_1]$

$$A+Bx \sim \left[\begin{array}{c|c} \begin{matrix} 1 & -x & 0 & \cdots & 0 & 0 \\ 0 & 1 & -x & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & & \vdots \\ 0 & & & \ddots & & 0 \\ 0 & \cdots & & 0 & 1 & -x \end{matrix} & M(x) \\ \hline 0 & N(x) \end{array}\right]$$

$L_{d \times (d+1)}$

(elimino i meni moltiplicando per blkdiag$(1,-1,1,-1,\dots)$)

Lemma: $A+Bx \sim \left[\begin{array}{c|c} L_{d \times d+1} & M(x) \\ \hline 0 & N(x) \end{array}\right] \sim \left[\begin{array}{c|c} L_{d \times d+1} & 0 \\ \hline 0 & N(x) \end{array}\right]$

A questo punto ripeto il ragionamento su $N(x)$: tre casi:

1) $N(x)v(x)=0 \rightsquigarrow$ ripeto

2) $w^T(x)N(x)=0 \rightsquigarrow$ ripeto su $N(x)^T$

3) $N(x)$ è regolare $\rightsquigarrow$ forma can. di Weierstrass

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Lemma 1: $[v_d, v_{d-1}, \dots v_0]$ e $[Av_d, \dots Av_1]$ sono lin. ind.

Mi basta dimostrarlo sugli Av_i: infatti se

$\alpha_0 v_0 + \alpha_1 v_1 + \dots + \alpha_d v_d = 0 \Rightarrow \alpha_1 A v_1 + \alpha_2 A v_2 + \dots + \alpha_d A v_d = 0$

(se la rel. di sx è non banale, lo è anche quella di dx, a meno che $\alpha_0 \neq 0, \alpha_1 = \alpha_2 = \dots = \alpha_d = 0$ $\Rightarrow v_0 = 0$ che contraddice scelta di $v(x)$ come pol. di grado minimo in $\mathrm{Ker}_{\mathbb{C}(x)} A + Bx$)

Supponiamo che esista una rel. di dip. lineare tra gli Av_i. Allora prendo

$\alpha_1 A v_1 + \dots + \alpha_d A v_d = 0$

$$w(x) = (\alpha_d + \alpha_{d-1} x + \dots + \alpha_1 x^{d-1})(v_0 + v_1 x + \dots + v_d x^d) =$$

$$= w_0 + w_1 x + \dots + w_d x^d + w_{d+1} x^{d+1} + \dots$$

$$A w_d = 0$$

$$(A+Bx)w(x) = (A+Bx)\alpha(x)v(x) = 0$$

$$\begin{bmatrix} A & & & & \\ B & A & & & \\ & B & A & & \\ & & \ddots & \ddots & \\ & & & B & A \\ & & & & B \end{bmatrix} \begin{bmatrix} w_0 \\ w_1 \\ \vdots \\ w_{d-1} \\ \hline w_d \\ \vdots \\ w_e \end{bmatrix} = 0$$

Visto che $Aw_d = 0$, allora anche

$$\begin{bmatrix} A & & & \\ B & A & & \\ & \ddots & \ddots & \\ & & B & A \\ & & & B \end{bmatrix} \begin{bmatrix} w_0 \\ w_1 \\ \vdots \\ w_{d-1} \end{bmatrix} = 0$$

- Take a basis that starts with the v_i; this block-triangularizes the pencil: $\begin{bmatrix} K(x) & L(x) \\ 0 & M(x) \end{bmatrix}$, where $K(x)$ is a Kronecker block.
- By the minimality of d, $M(x)$ is such that $d \times (d-1)$

$$\begin{bmatrix} M_0 & & & \\ M_1 & M_0 & & \\ & \ddots & \ddots & \\ & & M_1 & M_0 \\ & & & M_1 \end{bmatrix}$$

is nonsingular.

- Using this nonsingularity, one can prove that the system of Sylvester-like equations

$$\begin{bmatrix} I & E \\ 0 & I \end{bmatrix} \begin{bmatrix} K(x) & L(x) \\ 0 & M(x) \end{bmatrix} \begin{bmatrix} I & F \\ 0 & I \end{bmatrix} = \begin{bmatrix} K(x) & \color{red}{0} \\ 0 & M(x) \end{bmatrix}$$

is solvable (some work needed — details not given).

Kernel in $\mathbb{C}(x)$

The $(k \times (k+1))$ Kronecker blocks have kernel
$\begin{bmatrix} (-1)^k x^k & (-1)^{k-1} x^{k-1} & \dots & -x & 1 \end{bmatrix}^T$.
The other blocks have full column rank in $\mathbb{C}(x)$.

This can be used to characterize $\ker_{\mathbb{C}(x)}(A + Bx)$, using the fact that $\ker \operatorname{blkdiag}(C, D) = \operatorname{blkdiag}(\ker C, \ker D)$.

Remark: this gives a minimal basis, i.e., all other polynomial bases of $\ker_{\mathbb{C}(x)}(A + Bx)$ have higher degrees.

$$\underbrace{\begin{bmatrix} 1 & x & & & \\ & 1 & x & & \\ & & \ddots & \ddots & \\ & & & 1 & x \end{bmatrix}}_{k \times k+1} \begin{bmatrix} \pm x^k \\ \vdots \\ x^2 \\ -x \\ 1 \end{bmatrix} = 0$$

Se $A + Bx \sim \operatorname{blkdiag}(L_{k_1 \times k_1+1}, L_{k_2 \times k_2+1}, \dots L_{k_j \times k_j+1}, \dots$ altri blocchi non-$L)$

$$\ker\left[\begin{array}{c|cc|ccc|cc} 0 & 1 & x & & & & & \\ 0 & & & 1 & x & & & \\ \vdots & & & & 1 & x & & \\ & & & & & & 1 & x \\ 0 & & & & & & & 1 \end{array}\right] = \operatorname{span}\left[\begin{array}{c|c|c|c} 1 & & & \\ 0 & -x & & \\ 0 & 1 & x^2 & \\ \vdots & & -x & \\ 0 & & 1 & \\ & & & \end{array}\right] \qquad (*)$$

$(0, 1, 2)$ minimal indices (caratteristici dello spazio)

Ci sono altre basi: ad es.

$$\cdots \begin{bmatrix} 1 & 3x^3 & \\ 0 & -x & \\ 0 & 1 & x^5 \\ \vdots & & -x^4 \\ & & x^3 \\ & & 0 \\ 0 & & 0 \end{bmatrix} \cdots$$

Quella in cima, $(*)$, ha i gradi minori colonna per colonna (tra tutte le basi polinomiali) (stesso cosa funziona per il Kernel sinistro, vettori w t.c. $w^T(A+Bx)=0$.)