Example: square root

$$A = \begin{bmatrix} 4 & 1 & & \\ & 4 & 1 & \\ & & 4 & \\ & & & 0 \end{bmatrix}, \quad f(x) = \sqrt{x}$$

We look for an interpolating polynomial with

$$p(0) = 0,\ p(4) = 2,\ p'(4) = f'(4) = \frac{1}{4},\ p''(4) = f''(4) = -\frac{1}{32}.$$

I.e.,

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 4^3 & 4^2 & 4 & 1 \\ 3 \cdot 4^2 & 2 \cdot 4 & 1 & 0 \\ 6 \cdot 4 & 2 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_3 \\ p_2 \\ p_1 \\ p_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ \frac{1}{4} \\ -\frac{1}{32} \end{bmatrix},$$

$$p(x) = \frac{3}{256}x^3 - \frac{5}{32}x^2 + \frac{15}{16}x.$$

Example – continues

$$p(A) = \frac{3}{256}A^3 - \frac{5}{32}A^2 + \frac{15}{16}A = \begin{bmatrix} 2 & \frac{1}{4} & \frac{1}{64} & \\ & 2 & \frac{1}{4} & \\ & & 2 & \\ & & & 0 \end{bmatrix}.$$

(One can check that $f(A)^2 = A$.)

Example – square root

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad f(x) = \sqrt{x}$$

does not exist (because $f'(0)$ is not defined).

(Indeed, there is no matrix such that $X^2 = A$.)

Example – matrix exponential

$$A = S \begin{bmatrix} -1 & & & \\ & 0 & & \\ & & 1 & 1 \\ & & & 1 \end{bmatrix} S^{-1}, \quad f(x) = \exp(x).$$

$$\exp(A) = S \begin{bmatrix} e^{-1} & & & \\ & 1 & & \\ & & e & e \\ & & & e \end{bmatrix} S^{-1}$$

Can also be obtained as $I + A + \frac{1}{2}A^2 + \frac{1}{6}A^3 + \dots$
(not so obvious, for Jordan blocks. . .)

Example – matrix sign

$$A = S \begin{bmatrix} -3 & & & \\ & -2 & & \\ & & 1 & 1 \\ & & & 1 \end{bmatrix} S^{-1}, \quad f(x) = \operatorname{sign}(x) = \begin{cases} 1 & \operatorname{Re} x > 0, \\ -1 & \operatorname{Re} x < 0. \end{cases}$$

$$f(A) = S \begin{bmatrix} -1 & & & \\ & -1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} S^{-1}.$$

Not constant (for general S).

Instead, we can recover stable / unstable invariant subspaces of A as $\ker(f(A) \pm I)$.

If we found a way to compute $f(A)$ without diagonalizing, we could use it to compute eigenvalues via bisection...

Example – complex square root

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad f(x) = \sqrt{x}$$

We can play around with branches: let us say $f(i) = \frac{1}{\sqrt{2}}(1+i)$, $f(-i) = \frac{1}{\sqrt{2}}(1-i)$.

Polynomial: $p(x) = \frac{1}{\sqrt{2}}(1+x)$.

$$p(A) = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}.$$

(This is the so-called principal square root – we have chosen the values of $f(\pm i)$ in the right half-plane — other choices are possible).

(We get a non-real square root of A, if we choose non-conjugate values for $f(i)$ and $f(-i)$)

Example – nonprimary square root

With our definition, if we have

$$A = S \begin{bmatrix} 1 & & \\ & 1 & \\ & & 2 \end{bmatrix} S^{-1}, \quad f(x) = \sqrt{x}$$

we cannot get

$$f(A) = S \begin{bmatrix} 1 & & \\ & -1 & \\ & & \sqrt{2} \end{bmatrix} S^{-1} :$$

either $f(1) = 1$, or $f(1) = -1 \ldots$

This would also be a solution of $X^2 = A$, though.