Matrix polynomials

es. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} x^2 = \begin{bmatrix} 1+x+x^2 & x \\ x & 1+x \end{bmatrix}$

A matrix polynomial is $A(x) = A_0 + A_1 x + A_2 x^2 + \cdots + A_d x^d$.

può essere 0

We assume for now $A_i \in \mathbb{C}^{m \times m}$.

d "not exactly" degree: we admit zero leading coefficients.

Eigenvalues/vectors are pairs such that $A(\lambda)v = 0$.

If the polynomial is regular ($\det A(x)$ is not identically zero), then there is at most dm of them. They can be at ∞, like for pencils.

degree / grado

$\max\{i \le d : A_i \neq 0\}$ = degree, grado = d

$\det A(x)$ ha grado al più dm

$\Rightarrow$ al più dm autovalori

(molt. algebrica: quante volte compaiono)

$$A(x) = \begin{bmatrix} 1+x+x^2 & x \\ x & 1+x \end{bmatrix} \qquad \det A(x) = x^3 + \dots$$

1 autovalore a infinito

Remark: ci sono autoval. a infinito $\Leftrightarrow A_d$ singolare

(l'unico modo di ottenere un termine di grado dm nell'espansione è prendendo termini di A_d nell'espansione di Laplace $\Rightarrow$ il coefficiente di x^{dm} è proprio $\det A_d$

Reversal and infinite eigenvalues

Reversal of a matrix polynomial: same coefficients but in the opposite order:

$$\mathrm{Rev}(A_0 + A_1 x + A_2 x^2 + A_3 x^3) = A_3 + A_2 x + A_1 x^2 + A_0 x^3.$$

Lemma

Let $A(x)$ be a regular matrix polynomial. The eigenvalues of $\mathrm{Rev}\, A(x)$ are $\frac{1}{\lambda_i}$, where λ_i are the eigenvalues of $A(x)$. This includes also eigenvalues at ∞, with the convention that $\frac{1}{0} = \infty$ and $\frac{1}{\infty} = 0$.

Proof: direct verification for $\lambda \notin \{0, \infty\}$. Homogenize $\det A(x)$ to count eigvls at 0 and ∞.

Se $\det(A_0 + A_1\lambda + \dots + A_d\lambda^d) = 0$, allora

$$\det\left(\lambda^d\left(A_0\frac{1}{\lambda^d} + A_1\frac{1}{\lambda^{d-1}} + A_2\frac{1}{\lambda^{d-2}} + \dots + A_d\right)\right) = 0$$

$$\det\left(\lambda^d \cdot (\text{Rev}\,A)\left(\frac{1}{\lambda}\right)\right) = 0$$

$\Rightarrow$ Se $\lambda \neq 0$ è autoval. di $A(x)$, allora $\frac{1}{\lambda}$ autoval. di $\text{Rev}\,A(x)$

Funziona anche con $\frac{1}{0} = \infty$, $\frac{1}{\infty} = 0$

Omogeneizzando:

$\det(A_0 y^d + A_1 x y^{d-1} + \dots + A_d x^d) =$ poly. omogeneo di grado $d \cdot m$

molteplicità di 0 come autoval. = # di termini del tipo $y^{dm}, y^{dm-1}x, \dots y^{dm-k}x^k$ nulli molteplicità di ∞: numero di termini $x^{d_m}, x^{d_m-1}y, \dots x^{d_m-k}y^k$ che sono nulli

Se scambiate x, y questi concetti si scambiano

The companion linearization

Theorem

Let $A(x) = A_0 + A_1 x + A_2 x^2 + \cdots + A_d x^d$ be a matrix polynomial. Then, $\underline{\det A(x)} = \underline{\det C(x)}$, where $C(x)$ is the pencil ("Frobenius companion form")

$$C(x) = \begin{bmatrix} A_d x + A_{d-1} & A_{d-2} & \dots & +A_1 & +A_0 \\ I_m & -xI_m & & & \\ & I_m & -xI_m & & \\ & & \ddots & \ddots & \\ & & & I_m & -xI_m \end{bmatrix} \in \mathbb{C}[x]^{dm \times dm}.$$

We prove something stronger: there are $\underline{E(x)}, \underline{F(x)} \in \mathbb{C}(x)^{dm \times dm}$ with determinant 1 s.t. $E(x)C(x)F(x) = \underline{+}\operatorname{blkdiag}(A(x), I_{(d-1)m})$.

Proof (sketch): make linear combinations of columns to eliminate the blocks $-xI$.

$$C(x) = \begin{bmatrix} A_{d-1} & A_{d-2} & \cdots & A_0 \\ I & & & \\ & I & & \\ & & \ddots & \\ & & I & 0 \end{bmatrix} + \begin{bmatrix} A_d & & & \\ & -I & & \\ & & -I & \\ & & & \ddots \\ & & & & -I \end{bmatrix} x$$

Dimostriamo che esistono $E(x), F(x)$ con determinante 1 tali che

$$E(x)\,C(x)\,F(x) = \begin{bmatrix} A(x) & & & \\ & I_m & & \\ & & \ddots & \\ & & & I_m \end{bmatrix} \begin{matrix} \\ \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} d-1 \end{matrix}$$

$$\Rightarrow \det C(x) = \det A(x).$$

$$
\begin{bmatrix}
A_d x + A_{d-1} & A_{d-2} & \cdots & A_1 & A_0 \\
I & -xI & & & \\
& I & -xI & & \\
& & \ddots & \ddots & \\
& & & I & -xI
\end{bmatrix}
\cdot
\begin{bmatrix}
I & & & & \\
xI & I & & & \\
& & I & & \\
& & & \ddots & \\
& & & & I
\end{bmatrix}
=
$$

$$
= \begin{bmatrix}
A_d x + A_{d-1} & A_d x^2 + A_{d-1} x + A_{d-2} & A_{d-3} & \cdots & A_0 \\
I & 0 & & & \\
& I & -xI & & \\
& & \ddots & \ddots & \\
& & & I & -xI
\end{bmatrix}
\sim
\begin{bmatrix}
1 & & & \\
x & 1 & & \\
& & 1 & \\
& & & \ddots & \\
& & & & 1
\end{bmatrix}
$$

$$
\sim \begin{bmatrix}
A_d x + A_{d-1} & \cdots & A_d x^3 + A_{d-1} x^2 + A_{d-2} x + A_{d-3} & \cdots \\
I & 0 & 0 & \\
& I & 0 & \\
& & I & -xI \\
& & \ddots & \ddots
\end{bmatrix}
\sim
$$

$$\begin{bmatrix} * & * & * & \cdots & * & A(x) \\ I & & & & & \\ & I & & & & \\ & & I & & & \\ & & & \ddots & & \\ & & & & I & 0 \end{bmatrix}$$

d scambi colonne

$\sim$ determinante $(-1)^d$

$$\begin{bmatrix} A(x) & * & \cdots & * \\ & I & & \\ & & \ddots & \\ & & & I \end{bmatrix}$$

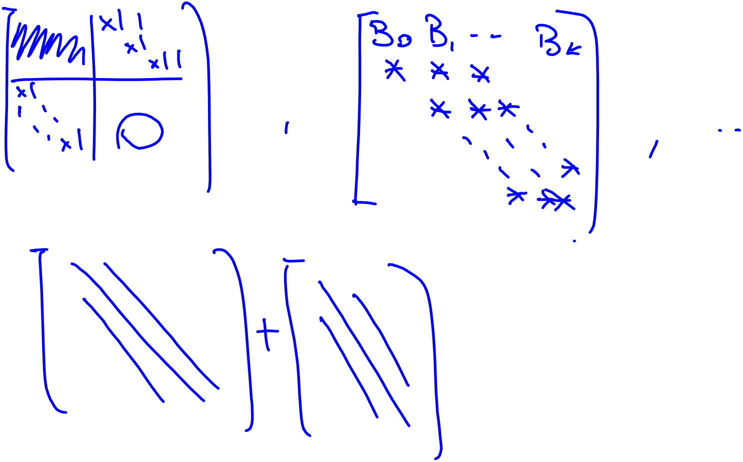
con determinante $\det A(x)$

(con operazioni su righe posso far sparire anche gli "$*$").

Other linearizations

Other pencils with the same property $E(x)C(x)F(x) = \mathrm{blkdiag}(A(x), I_{(d-1)m})$ can be constructed — they are called linearizations.

Some final projects available on methods to construct them.



Eigenvector recovery

Theorem

$v \neq 0$ is an eigenvector of $A(x)$ (with eigvl $\lambda \neq \infty$) iff

$$w(\lambda, v) = \begin{bmatrix} \lambda^{d-1} v \\ \lambda^{d-2} v \\ \vdots \\ v \end{bmatrix}$$

is an eigenvector of $C(x)$.

Proof: direct verification. Start calling v the last block of $w \ldots$.

Sia $\underset{\neq 0}{w} = \begin{bmatrix} * \\ * \\ * \\ * \\ y \end{bmatrix}$ un autovettore di $C(x)$

$$\begin{bmatrix} A_d\lambda + A_{d-1} & \cdots & A_0 \\ I & -\lambda I & \\ & \ddots & \ddots \\ & I & -\lambda I \\ & & I & -\lambda I \end{bmatrix} \begin{bmatrix} \\ \\ \lambda^2 y \\ \lambda y \\ y \end{bmatrix} = 0$$

dev'essere λy perché l'ultima riga del prodotto faccia 0

dev'essere $\lambda^2 y$ perché la penultima riga faccia 0

... arrivo a dire che

$$\begin{bmatrix} A_d\lambda + A_{d-1} & A_{d-2} & \cdots & A_1 & A_0 \\ I & -\lambda I & & & \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ & & & I & -\lambda I \end{bmatrix} \begin{bmatrix} \lambda^{d-1} y \\ \vdots \\ \lambda^2 y \\ \lambda y \\ y \end{bmatrix} = 0$$

La prima riga del prodotto mi dice che

$A(\lambda)y = 0 \Rightarrow$ y autovettore di A

(y non può essere 0, altrimenti $w=0$)

(L'altra freccia è solo una verifica)

Eigenvectors are not independent

Small surprise: the eigenvectors of $A(x)$ are (usually) not linearly independent.
(How could they be? There are too many of them...)
However, $w(\lambda_i, v_i)$ are linearly independent.

Un polinomio $m \times m$ ha dm autovettori in $\mathbb{C}^m$
Sono troppi...

Ex

$$A(x) = \begin{bmatrix} (x-1)(x-2) & 0 \\ 0 & (x-3)(x-4) \end{bmatrix}$$

autoval/vett:

$$\lambda = 1, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \lambda = 2, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \lambda = 3, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \lambda = 4, \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Però sono lin. indipendenti

$$\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ 0 \\ 1 \end{bmatrix}$$

(perché sono gli autovettori di $C(x)$)

Application

What do we use linearization / eigenvalues of matrix polynomials for?

Linear differential equations: (assume $\det(A_2) \neq 0$)

$$A_2\ddot{x} + A_1\dot{x} + A_0 x = 0, \quad x : [t_0, t_f] \to \mathbb{R}^n. \qquad \text{(ode)}$$

$x = x(t)$

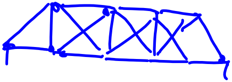
Special solutions: $e^{\lambda t}v$, where (λ, v) eigenpair of the matrix polynomial.

General solution: via linearization of $A_2x^2 + A_1x + A_0$ (matrix exponential of $-C_1^{-1}C_0$).

(Stable solutions: invariant subspace formed by eigenvalues with $\operatorname{Re}\lambda < 0$.)

What about polynomials with eigenvalues at ∞ / singularities? More involved — differential-algebraic equations. Example $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$.

$$A_2 \ddot{y}_2 + A_1 \dot{y} + A_0 y = 0 \qquad (*)$$



$$\underset{\sim}{A_1} \dot{y} + A_0 y = 0$$

$$A(x) = A_2 x^2 + A_1 x + A_0$$

Autoval / vett $\iff$ soluzioni particolari

$$A(\lambda) v = 0 \iff y(t) = e^{\lambda t} v \text{ soluzione di } (*)$$

Ad es., se ho $2m$ autoval. distinti con $\mathrm{Re}\,\lambda < 0$, le soluzioni sono comb. lineari di queste sol. particolari, e vanno tutte a zero per $t \to \infty$, quella che va a zero più piano ha $\mathrm{Re}(\lambda)$ massimo...

Se $\det A_2 \neq 0$, posso trovare una sol. generale linearizzando: definisco $z := \dot{y}$, pongo

$$\begin{bmatrix} -A_2 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \dot{z} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} A_1 & A_0 \\ I & 0 \end{bmatrix} \begin{bmatrix} z \\ y \end{bmatrix}$$

Frobenius companion linearization

$$\begin{bmatrix} \dot{z} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -A_2 & 0 \\ 0 & I \end{bmatrix}^{-1} \begin{bmatrix} A_1 & A_0 \\ I & 0 \end{bmatrix} \begin{bmatrix} z \\ y \end{bmatrix}$$

$$\begin{bmatrix} z(t) \\ y(t) \end{bmatrix} = \operatorname{expm}\left(t \cdot \begin{bmatrix} -A_2 & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} A_1 & A_0 \\ 1 & 0 \end{bmatrix} \right) \begin{bmatrix} z_0 \\ y_0 \end{bmatrix}.$$

Esempi in cui il coeffic. di testa è singolare:
Supponiamo di avere già un'equazione del primo ordine (via linearizzazione)

$\times \quad C_1 \dot{u} + C_0 u = 0$

Posso mettere $C_1 x + C_0$ in forma di Kronecker:

$E(C_1 x + C_0)F = \mathrm{blkdiag}(\dots)$

Produce un sistema di equazioni equivalente

$$E C_1 F \dot{v} + E C_0 F v = 0 \qquad v(t) := F^{-1} u(t)$$

blkdiag $\to$ equazioni separate

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{v}_1 \\ \dot{v}_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0 \quad \to$$

$$\begin{cases} \dot{v}_2 + v_1 = 0 \\ 0 + v_2 = 0 \end{cases}$$

Il problema ai valori iniziali collegato a quest'equazione ha soluzione solo se $v_2(0) = 0$ e $v_1(0) = 0$

La soluzione è $v_2(t) \equiv 0$ $\quad v_1(t) \equiv 0$

DAE. differential-algebraic equations

Jordan chains [Gohberg-Lancaster-Rodman book, Sec. 1.4]

One can define Jordan chains of a matrix polynomial using derivatives:

$(A + \lambda B)v_1 + B v_0 = 0$

$$
\begin{aligned}
&A(\lambda)v_0 = 0, \\
&A'(\lambda)v_0 + A(\lambda)v_1 = 0, \\
&\frac{1}{2}A''(\lambda)v_0 + A'(\lambda)v_1 + A(\lambda)v_2 = 0 \\
&\vdots
\end{aligned}
$$

With this definition, $\boxed{v_0 e^{\lambda t}}$, $\boxed{(v_0 t + v_1)e^{\lambda t}}$ $(v_0 t^2 + v_1 t + v_2)e^{\lambda t}$, ... are special solutions of (ode).

How to define Jordan chains at ∞? As Jordan chains at zero of $\operatorname{Rev} A(x)$. Def: Diciamo che $A(x)$ ha una catena di Jordan $v_0, v_1, \dots v_k$ a $\lambda=\infty$ se $\operatorname{rev} A(x)$ ha una catena di Jordan $v_0, v_1, \dots v_k$ a $\lambda=0$.

$$P(x) = P(\lambda) + P'(\lambda)(x-\lambda) + \frac{1}{2}P''(\lambda)(x-\lambda)^2 + \dots$$

$$v(x) = v_0 + v_1(x-\lambda) + v_2(x-\lambda)^2 + \dots$$

$$P(x)\,v(x) = P(\lambda)v_0 + \left(P'(\lambda)v_0 + P(\lambda)v_1\right)(x-\lambda) +$$
$$+ \left(\frac{1}{2}P''(\lambda)v_0 + P'(\lambda)v_1 + P(\lambda)v_2\right)(x-\lambda)^2 + \dots$$

Catena di Jordan lunga k $\Leftrightarrow$ esiste $v(x)$ tale che $P(x)v(x)$ ha i primi k termini dello sviluppo come polinomio in $(x-\lambda)$ nulli

$$\Leftrightarrow \; P(x)v(x) = (x-\lambda)^k \cdot w(x)$$

The problem with linearizations and $\lambda = \infty$

The relation $C(x) \sim D(x)$ iff $C(x) = E(x)D(x)F(x)$ preserves the sizes of Jordan chains at $\lambda \in \mathbb{C}$ (why?), but not of those at infinity. This can be seen already for degree 1: the pencils

$$D(x) = I + 0x = \begin{bmatrix} 1 & \\ & 1 \end{bmatrix}$$

and

$$C(x) = \begin{bmatrix} 1 & x \\ & 1 \end{bmatrix} = D(x)F(x) \quad \text{with } F(x) = \begin{bmatrix} 1 & x \\ & 1 \end{bmatrix}$$

have different Jordan structures at ∞.

$A(x) = A_0 + A_1 x$ Linearizzazione: pencil $L(x)$ tale che

$$E(x) L(x) F(x) = \mathrm{blkdiag}(A(x), I_{(d-1)n})$$

$E(x) \in \mathbb{C}[x]^{m \times m}$, $F(x) \in \mathbb{C}[x]^{m \times m}$, $\det E(x) = \det F(x) = \pm 1$

$$A(x) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} x$$

$\det A(x) = 1$ no autovalore ∞ con m.a. 2

Forma di Kronecker: $\begin{bmatrix} \boxed{1} & 0 \\ 0 & \boxed{1} \end{bmatrix}$ 2x blocco di Jordan a ∞ di dimensione 1

$$A(x) F(x) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \underbrace{\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}}_{\det = 1} = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}$$

blocco di Jordan a ∞ di dimensione 2

Invece, pre/post moltiplicare per matrici con det $= 1$ non cambia le dimensioni dei blocchi di J. finiti

blocco di J. finito $\Leftrightarrow$

$$P(\lambda)\, v_0 = 0$$

$$P(\lambda)\, v_1 + P'(\lambda)\, v_0 = 0$$

$$P(\lambda)\, v_2 + P'(\lambda)\, v_1 + \frac{1}{2} P''(\lambda)\, v_0 = 0$$

$$\vdots$$

E queste relazioni restano valide (per nuovi v_i) anche se pre/post moltiplico per $E(x)$, $F(x)$:

$$E(\lambda) P(\lambda) F(\lambda)\, v_0 = 0$$

$$E(\lambda) P(\lambda) F(\lambda)\, v_1 +$$

Teo: Le catene di Jordan di $C(x) \in \mathbb{C}[x]^{m \times n}$ e quelle di $B(x) = E(x)\, C(x) F(x)$ hanno la stessa lunghezza, per ogni $E(x) \in \mathbb{C}[x]^{m \times m}$, $\boxed{\text{per } \lambda \neq \infty}$ $F(x) \in \mathbb{C}[x]^{n \times n}$ con $\det E(x) = \det F(x) = 1$.

dim:

Nota che $F(x)^{-1}$ è un polinomio: infatti,

$$F(x)^{-1} = \frac{1}{\det F(x)} \cdot \mathrm{Adj} F(x)^T \qquad \mathrm{Adj} F(x)_{ij} = \det \begin{pmatrix} F(x) \text{ senza la} \\ \text{riga } i \text{ e la colonna } j \end{pmatrix} \cdot (-1)^{i+j}$$

Se $C(x)$ ha una catena di J. lunga k,

allora $C(x) v(x) = (x-\lambda)^k w(x)$, $w(x) \in \mathbb{C}[x]^n$ $w(\lambda) \neq 0$

$$E(x) C(x) F(x)\, F(x)^{-1} v(x) = (x-\lambda)^k E(x) w(x)$$

$$\Rightarrow B(x)S(x) = (x-\lambda)^k t(x)$$

polinomi con $t(\lambda) \neq 0$

Se scrivo $s(x) = s_0 + s_1(x-\lambda) + s_2(x-\lambda)^2 + \dots$, allora $s_0, s_1, \dots s_{k-1}$ sono una catena di Jordan di $B(x)$

$$s_0 = \underbrace{F(\lambda)}^{-1} v_0$$

$$s(x) = \left(\square + \square\,(x-\lambda) + \square\,(x-\lambda)^2 + \dots\right)\left(v_0 + v_1(x-\lambda) + v_2(x-\lambda)^2 + \dots\right)$$

In part. per una linearizzazione

$$E(x)C(x)F(x) = \mathrm{blkdiag}(A(x), I)$$

Le catene di Jordan di $\mathrm{blkdiag}(A(x), I)$ sono del tipo $\begin{bmatrix} v_0 \\ 0 \end{bmatrix}, \dots \begin{bmatrix} v_k \\ 0 \end{bmatrix}$

Strong linearizations

$L(x) = L_0 + L_1 x \quad \mathrm{Rev}\,L(x) = L_1 + L_0 x$

A linearization is said to be strong if Rev $L(x)$ is also a linearization of Rev $A(x)$.

$\iff$ se esistono $\hat{E}(x), \hat{F}(x)$ t.c. $\hat{E}(x)\,\mathrm{rev}L(x)\hat{F}(x) = \mathrm{blkdiag}(\mathrm{Rev}A(x), I)$

Result

The companion pencil $C(x)$ is a strong linearization.

Proof: again linear combinations of columns to 'fold up' the polynomial Rev $A(x)$.

$\to$ Le lunghezze dei blocchi di Jordan a 0 di rev $A(x)$ sono le stesse di rev $C(x)$ $\iff$

Le lunghezze dei blocchi a ∞ di $A(x)$ sono le stesse di $C(x)$

$$\begin{bmatrix} \tilde{A}_d + xA_{d-1} & A_{d-2}x & A_{d-3}x & \cdots & A_0x \\ xI & -I & & & \\ & xI & & & \\ & & \ddots & & \\ & & & xI & -I \end{bmatrix} \sim$$

$$\begin{bmatrix} \cdots & A_2x & A_1x + A_0x^2 & A_0x \\ \ddots & \ddots & & \\ xI & -I & & \\ & 0 & -I & \end{bmatrix} \sim$$

$$
\begin{bmatrix}
\cdots & & A_2x+A_1x^2+A_0x^3 & A_1x+A_0x^2 & A_0x \\
 & \ddots & & & \vdots \\
\ddots & & & & \\
 & xI & -I & & \\
\ddots & & 0 & -I & \\
 & & & & -I
\end{bmatrix}
\sim
$$

$$\begin{bmatrix} A_d + A_{d-1}x + A_{d-2}x^2 + \dots + A_0 x^d & * & * & * & * \\ & -I & & & \\ & & -I & & \\ 0 & & & \ddots & \\ & & & & -I \end{bmatrix} \sim$$

(via comb. lineari di righe)

$$\sim \begin{bmatrix} \operatorname{Rev} A(x) & 0 & 0 & \cdots & 0 \\ & -I & & & \\ & & -I & & \\ & & & \ddots & \\ & & & & -I \end{bmatrix}$$

Smith form [Gohberg–Lancaster–Rodman book, appendix S1]

Quick review of other invariants for matrix polynomials...

Smith normal form

There are matrices $\underline{E(x)}$, $\underline{F(x)}$ with determinant 1 such that $E(x)A(x)F(x) = \mathrm{diag}(d_1(x), d_2(x), \dots, d_r(x), 0, 0, \dots, 0)$, and $d_i(x) \mid d_{i+1}(x)$ for all i.

Actually an algebra result — holds in every PID. (Proof idea: a sort of back-and-forth Gaussian elimination like the one used to compute inverses. Instead of division, use Bézout identities).
The d_is are uniquely defined (GCDs of all $i \times i$ minors).
Reveals:

- Rank over $\mathbb{C}(x)$;
- Eigenvalues (roots of $d_r(x)$);
- Sizes of Jordan chains (depend on how many $d_i(\lambda)$ vanish).

Esistono $E(x), F(x)$ t.c.

$$E(x)A(x)F(x) = \begin{bmatrix} d_1(x) & & & & & & \\ & d_2(x) & & & & \text{\Large 0} & \\ & & \ddots & & & & \\ & & & d_r(x) & & & \\ & & & & 0 & & \\ & \text{\Large 0} & & & & \ddots & \\ & & & & & & 0 \end{bmatrix}$$

$d_i(x) \mid d_{i+1}(x)$ $\qquad$ $r =$ rango di $A(x)$ (su $\mathbb{C}(x)$)

es.
$$\begin{bmatrix} x & & & & & \\ & x(x+1) & & & \text{\Large 0} & \\ & & x^2(x+1)(x+2) & & & \\ & & & 0 & & \\ & \text{\Large 0} & & & \ddots & \\ & & & & & 0 \end{bmatrix}$$

Idea dim: se fossimo su un campo,

$$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix} \longrightarrow \begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

su un anello:

$$\begin{bmatrix} a_{11}(x) & * & * \\ a_{21}(x) & * & * \\ a_{31}(x) & * & * \end{bmatrix} \longrightarrow \begin{bmatrix} \mathrm{mcd}(a_{11}, a_{21}, a_{31}) & * & * \\ 0 & * & * \\ 0 & * & * \end{bmatrix}$$

Minimal indices [Forney, '72]

Quick review of other invariants for matrix polynomials...
The generalization of sizes of singular Kronecker blocks are minimal indices.

Like in the degree-1 case, $\ker_{\mathbb{C}(x)} A(x)$ and $\ker_{\mathbb{C}(x)} A^T(x)$ admit polynomial bases with degrees "as small as possible" — these degrees are called minimal indices.

Linearizations do not always preserve minimal indices — but sometimes they change them predictably. For instance, for $C(x)$, right minimal indices are preserved, left minimal indices are increased by $d-1$.