Lyapunov equations

pos. semidefinite

Lyapunov equation

$$\boxed{A^*X + XA + Q = 0,} \quad Q = Q^* \succeq 0. \qquad \text{(L)}$$

Special case of the Sylvester equation.

Lemma

Suppose (L) has a unique solution X; then X is symmetric.

Proof: transpose everything; X^* is another solution.

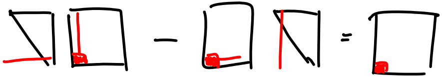
$(L)^*$: $X^*A + A^*X^* + Q = 0$

(Come risolveremo $AX - XB = C$? Prenderemo cambi di base $A = Q_A T_A Q_A^*$ $\quad B = Q_B T_B Q_B^*$,

$$Q_A^* Q_A T_A Q_A^* X Q_B - Q_A^* X Q_B T_B Q_B^* Q_B = Q_A^* C Q_B$$

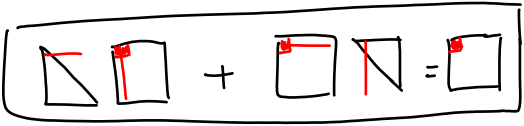
$$\Rightarrow \quad T_A \hat{X} - \hat{X} T_B = \hat{C}, \qquad \hat{X} = Q_A^* X Q_B, \quad \hat{C} = Q_A^* C Q_B$$

che risolviamo per sostituzione, un elemento di $\hat{X}$ per volta a partire da $\hat{X}_{n,1}$:



Nel caso Lyapunov, basta <u>una</u> forma di Schur: $A = Q_A T_A Q_A^*$, che produce

$$T_A^* \hat{X} + \hat{X} T_A = \hat{Q}$$



serve che $\lambda_i + \lambda_j^* \neq 0$ $\forall i,j$, dove λ_i autoval. di A

che posso risolvere per sostituzione a partire da $\hat{X}_{1,1}$.

Altra osservazione: (L) ha soluz. unica se e solo se $\lambda_i + \lambda_j^* \neq 0$ $\forall i,j$ (λ_i autoval. di A), in particolare se $\Lambda(A) \subseteq LHP$ o $\Lambda(A) \subseteq RHP$

Lyapunov equation: positivity

$\mathrm{Re}\,\lambda_i < 0$

1

Lemma

Suppose A has eigenvalues in the (open) LHP. Then, $Q \succeq 0$ implies $X \succeq 0$, and $Q \succ 0$ implies $X \succ 0$.

Proof Check that

$$\boxed{X = \int_0^\infty e^{A^* t} Q e^{At} \, \mathrm{d}t}$$

(via $\frac{\mathrm{d}}{\mathrm{dt}} e^{A^* t} Q e^{At} = \dots$).

2

Lemma

Suppose $\underline{Q \succ 0}$ and $\underline{X \succ 0}$. Then, A has eigenvalues in the (open) LHP.

Proof Let $Av = \lambda v$; then $0 < v^* Q v = \dots$

Alternative statement: A has all its eigenvalues in the LHP if there exists $X \succ 0$ such that $A^* X + XA \prec 0$.

① $A^* X + XA + Q = 0$ <u>Lemma</u>: se $\Lambda(A) \subseteq LHP$,

$$X = \int_0^\infty e^{tA^*} Q \, e^{tA} \, dt$$

(noto che è necessario che $\Lambda(A) \subseteq LHP$ perché l'integrale converge per ogni Q)

(se $\Lambda(A) \subseteq LHP$, $\exp(tA) \to 0$ esponenzialmente)

(si vede passando a blocchi di Jordan: $A = VJV^{-1}$, $\exp(tA) = V \exp(tJ) V^{-1}$ e su ogni blocco J_i $\exp(tJ_i) \to 0$ per $t \to \infty$, per es. $J_i = \boxed{\lambda}$ produce $\exp(t\lambda) \to 0$ se $\operatorname{Re} \lambda < 0$)

$$A^* \int_0^\infty \exp(tA^*)\, Q \exp(tA)\, dt + \int_0^\infty \exp(tA^*)\, Q \exp(tA)\, dt\, A =$$

$$= \int_0^\infty \underbrace{\exp(tA^*) A^* Q \exp(tA) + \exp(tA^*)\, Q A \exp(tA)}\, dt =$$

$$= \underbrace{\Big[\exp(tA^*)\, Q \exp(tA) \Big]_0^\infty = 0 - Q}_{0} \qquad \text{(formula int)} \quad \square$$

Se $Q \geqslant 0$, $(\exp(tA))^* Q \exp(tA) \geqslant 0$, allora

$$\int_0^\infty \exp(tA^*)\, Q \exp(tA) \geqslant 0$$

② $X > 0, \; Q > 0 \quad A^*X + XA + Q = 0 \Rightarrow \Lambda(A) \subseteq LHP$

<u>dim</u>: $Av = \lambda v$

$$0 = v^*(A^*X + XA + Q)v = \bar{\lambda}\, v^*Xv + v^*Xv\lambda + v^*Qv =$$

$$= (\bar{\lambda} + \lambda)\, v^*Xv + v^*Qv$$

$$\Rightarrow \bar{\lambda} + \lambda = -\frac{v^*Qv}{v^*Xv} < 0$$

$\Updownarrow$

$a - \cancel{b}i + a + \cancel{b}i < 0$

statement alternativo: se trovo $X > 0$ tale che $A^*X + XA < 0$, allora $\Lambda(A) \subseteq LHP$

Lyapunov's version

$x(t) = \exp(tA) \cdot x_0$

Alternative way to see it: the dynamical system $\boxed{\dot{x}(t) = Ax(t)}$ is stable (i.e., $\lim_{t\to\infty} x(t) = 0$) for all initial values if and only if there exists $X \succ 0$ such that $A^*X + XA \prec 0$.

Lyapunov's proof: $E(t) = x(t)^* X x(t)$ ('energy function') is such that $\dot{E}(t) \leq 0$.

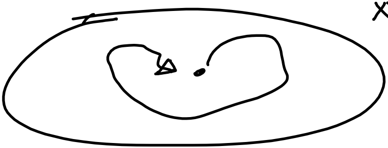
"strongly asymptotically stable"

Funzione energia $E(t) = x^* X x$:

$\dot{E} = \dot{\overline{x^* X x}} = \dot{x}^* X x + x^* X \dot{x} = x^* (A^* X + XA) x < 0$

$< (x^* x) \cdot \lambda_{min}(A^* X + XA)$

$E(t) \to 0 \Rightarrow$ se partiamo con x t.c. $x^* X x <$ cost



$x^* X x < \text{const}$

Discrete-time version

Many of these results also come in a 'discrete-time' variant; in this case:

Stein's equation

A has all its eigenvalues in the (open) unit disc iff

$$X - A^*XA = Q, \quad Q \succ 0$$

has a solution $X \succ 0$.

Deals with stability of the discrete-time system $x_{t+1} = Ax_t$.

Discrete-time version

Discrete-time version of the integral formula:

If A has all its eigenvalues inside the (open) unit disc then

$$X = \sum_{k=0}^{\infty} (A^*)^k Q A^k.$$

Proof $(I - A^T \otimes A^*)\,\mathrm{vec}(X) = \mathrm{vec}(Q)$, series for the inverse.