Controllability

Definition

$(A, B) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m}$ is controllabile iff $K(A, B) = \mathbb{R}^n$, where

$$K(A, B) := \operatorname{span}(B, AB, A^2B, \dots).$$

(Rmk: basta fermarsi a A^nB, perché $A^nB = (\alpha_0 I + \alpha_1 A + \dots + \alpha_{n-1}A^{n-1})B$, dove $\alpha_0 I + \alpha_1 A + \dots + \alpha_{n-1}A^{n-1} - A^n$ è il pol. caratter. di A.

Lemma

There exists a nonsingular $M \in \mathbb{R}^{n\times n}$ such that

$$M^{-1}AM = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline 0 & A_{22} \end{array}\right], \quad M^{-1}B = \left[\begin{array}{c} B_1 \\ \hline 0 \end{array}\right]$$

(with $A_{11} \in \mathbb{R}^{n_1\times n_1}$, $A_{22} \in \mathbb{R}^{n_2\times n_2}$, $B_1 \in \mathbb{R}^{n_1\times m}$, and $n_2 \neq 0$) if and only if (A, B) is not controllable.

$$M^{-1}AM = \begin{array}{cc} n_1 & n_2 \\ \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline 0 & A_{22} \end{array}\right] & \end{array} , \quad M^{-1}B = \begin{array}{c c} \left[\begin{array}{c} B_1 \\ \hline 0 \end{array}\right] & \begin{array}{c} n_1 \\ n_2 \end{array} \end{array}$$

$$M = \begin{array}{cc} \left[\begin{array}{c|c} M_1 & M_2 \end{array}\right] \\ \quad n_1 \quad\; n_2 \end{array}$$

$$M^{-1}A^kB = (M^{-1}A\,M)^k\, M^{-1}B = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}^k \begin{pmatrix} B_1 \\ 0 \end{pmatrix} = \begin{bmatrix} A_{11}^k & * \\ 0 & A_{22}^k \end{bmatrix}\begin{bmatrix} B_1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} A_{11}^k B_1 \\ 0 \end{bmatrix}$$

$$A^kB = M\begin{bmatrix} A_{11}^k B \\ 0 \end{bmatrix} = \begin{bmatrix} M_1 & M_2 \end{bmatrix}\begin{bmatrix} A_{11}^k B \\ 0 \end{bmatrix} = M_1 A_{11}^k B$$

$$\boxed{M_1}\,\boxed{A}^k\,\boxed{B}$$

Le colonne di A^kB stanno in $\operatorname{Im} M_1$

$\Rightarrow \operatorname{span}(B, AB, \dots) \subseteq \operatorname{Im} M_1 \neq \mathbb{R}^n$

(Idea: $K(A,B) = \mathrm{span}(B, AB, \ldots)$ è A-invariante: se v è comb. lin. di $B, AB, \ldots$, allora anche Av lo è)

Sia M_1 una matrice le cui colonne sono una base di $K(A,B)$, e la completo a una matrice invertibile $M = [M_1 \ M_2]$

Allora,

1) $$M^{-1}B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}$$ (perché le colonne di B sono comb. lin. di quelle di M_1

(se $M v_i = B_i$, allora $v_i = \begin{bmatrix} * \\ 0 \end{bmatrix}$))

2) $$M^{-1}AM = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$$: difatti $AM_1 = M_1 A_{11} + M_2 \cdot 0$, quindi
$$A[M_1 \ M_2] = [M_1 \ M_2] \begin{bmatrix} A_{11} & * \\ 0 & * \end{bmatrix}.$$

Caso single-input $(m=1)$:

$$K(A,b) = \text{limite degli spazi di Krylov}$$

M_1 è lo spazio generato dal metodo di Arnoldi quando si ha breakdown (spazio di Krylov).

$K(A,b) =$ (più piccolo sottospazio A-invariante che contiene b)

Proof

$\Rightarrow$ Partition $M = \begin{bmatrix} M_1 & M_2 \end{bmatrix}$ conformably. Then,

$A^k B = M \begin{bmatrix} A_{11}^k B_1 \\ 0 \end{bmatrix} = M_1 A_{11}^k B_1$, so $K(A, B) \subseteq \operatorname{Im} M_1$.

$\Leftarrow$ Let the columns of M_1 be a basis of $K(A, B)$, and complete it to a nonsingular $M = \begin{bmatrix} M_1 & M_2 \end{bmatrix}$. Then, $M^{-1}AM$ is block triangular (because M_1 is A-invariant), and $M^{-1}B$ has zeros in the second block row (because the columns of B lie in $\operatorname{Im} M_1$).

(Linear algebra characterization: $K(A, B)$ is the smallest A-invariant subspace that contains B. It's the space Q_n that we obtain after we encounter breakdown in Arnoldi.)

Kalman decomposition

Kalman decomposition

For every matrix pair $(A, B) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m}$, there is a change of basis M such that

$$M^{-1}AM = \begin{array}{cc} n_1 & n_2 \\ \end{array} \hspace{-6em} \begin{array}{c} \\ \begin{bmatrix} A_{11} & A_{12} \\ \boxed{0} & A_{22} \end{bmatrix} \end{array}, \quad M^{-1}B = \begin{array}{cc} \begin{bmatrix} B_1 \\ \boxed{0} \end{bmatrix} & \begin{array}{c} n_1 \\ n_2 \end{array} \end{array},$$

with (A_{11}, B_1) controllable.

Proof: as above: take M_1 such that its columns are a basis of the 'controllable space' $K(A, B)$, then complete it to a basis of $\mathbb{R}^n$.

M_1 è una base dello spazio $\operatorname{span}(M_1B_1, M_1A_{11}B_1, M_1A_{11}^2B_1, \ldots)$

$\underbrace{\operatorname{span} M_1}_{\dim = n_1} = M_1 \cdot \operatorname{span}(B_1, A_{11}B_1, A_{11}^2B_1, \ldots)$ $\leftarrow$ deve avere dim. n_1

$\Rightarrow \operatorname{span}(B_1, A_{11}B_1, A_{11}^2B_1)$ deve avere dimensione n_1

Stabilizability

Definition

(A, B) is stabilizable if in its Kalman decomposition A_{22} is stable (i.e., $\Lambda(A_{22}) \subseteq LHP$).

Note that this definition is well-posed even if M is non-unique: the eigenvalues of A_{11} are the eigenvalues of $A|_{K(A,B)}$, and those of A_{22} are the remaining eigenvalues of A (counting with their algebraic multiplicity).

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{bmatrix} A_{11}x_1 + A_{12}x_2 + B_1 u \\ A_{22}x_2 \end{bmatrix} \begin{array}{l} \leftarrow \text{controllabile} \\ \leftarrow \text{già stabile} \end{array}$$

Controllability Lyapunov equation

Theorem

If A is a stable matrix and $\boxed{(A, B) \text{ is controllable}}$, then the solution of $\boxed{AX + XA^* + BB^* = 0}$ is positive definite.

Proof We know already that $X = \boxed{\int_0^\infty \exp(At)BB^* \exp(A^*t)\,\mathrm{d}t \succeq 0.}$

Let $v \neq 0$ be any vector. We need to show that $v^*Xv \neq 0$.

If $v^* \exp(tA)B \neq 0$ for some t, then we are done (it is nonzero in a neighbourhood by continuity...).

If $v^* \exp(tA)B = 0$ for all t, then $v^*B = 0$ (taking $t = 0$),

$A = \lim\limits_{t \to 0} \dfrac{\exp(tA) - I}{t}$

$$\lim_{t \to 0} \frac{1}{t} v^*(\exp(tA) - I)B = v^*AB = 0,$$

$$\lim_{t \to 0} \frac{1}{t^2} v^*(\exp(tA) - I - tA)B = v^*\frac{1}{2}A^2B = 0,$$

$$\vdots$$

Se $v^* X v = v^* \left(\int_0^\infty \exp(tA) B \cdot B^* \exp(tA^*)\, dt \right) v =$

$$\int_0^\infty \left[v^* \exp(tA) B \right] \left[v^* \exp(tA) \cdot B \right]^* dt = \int_0^\infty \| v^* \exp(tA) B \|^2 dt$$

Se trovo t tale che $v^* \exp(tA) B \neq 0$, trovo anche un intorno in cui è $\neq 0$, quindi l'integrale è strettamente positivo.

Se $v^* \exp(tA) B = 0 \ \forall t \geq 0$, allora voglio dire che $v^* B = v^* AB = v^* A^2 B = \ldots = 0$, e quindi (A, B) non è controllabile (perché $K(A,B) \subseteq \operatorname{Ker} v^* \neq \mathbb{R}^n$).

Remark: Il contrario è abbastanza diverso:

se (cambiando base) $A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$, $B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}$,

allora $\exp(tA)B = \left(\sum_{k=0}^{\infty} \frac{t^k}{k!} A^k\right) B = \sum_{k=0}^{\infty} \frac{t^k}{k!} \begin{bmatrix} A_{11}^k B_1 \\ 0 \end{bmatrix} = \begin{bmatrix} * \\ 0 \end{bmatrix}$