Optimal control

Bass: fornisce F t.c. $A+BF$ è stabile
Non è l'unica (ad es. piccole perturbaz. vanno bene)

Several choices for stabilizing — for instance, you can choose different αs in Bass's algorithm.

Is there an 'optimal' one?

Linear-quadratic optimal control

Find $u : [0, \infty] \to \mathbb{R}$ (piecewise C^0, let's say) that minimizes

$u = F \cdot x$

$$E = \int_0^\infty x^T Q x + u^T R u \, dt =: E(u)$$

$$\text{s.t. } \dot{x} = Ax + Bu, \; x(0) = x_0.$$

Minimum 'energy' defined by a quadratic form ($R \succeq 0$, $Q \succeq 0$).

We assume $R \succ 0$: control is never free. Trickier problem otherwise.

Se u_* ottimo, per ogni funzione v e per ogni reale h

$E(u_* + hv) \geq E(u_*)$ $\quad \frac{d}{dh} E(u_* + hv) = 0$

Optimal control — solution

Using calculus of variations tools, one can prove that (Pontryagin's maximum principle)

A pair of functions u, x solves the optimal control problem iff there exists a function $\mu(t)$ ('Lagrange multiplier') such that

DAE

$$\begin{bmatrix} 0 & I & 0 \\ -I & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot\mu \\ \dot x \\ \dot u \end{bmatrix} = \begin{bmatrix} 0 & A & B \\ A^T & Q & 0 \\ B^T & 0 & R \end{bmatrix} \begin{bmatrix} \mu \\ x \\ u \end{bmatrix},$$

$x(0) = x_0$, $\lim_{t\to\infty} \begin{bmatrix} \mu \\ x \\ u \end{bmatrix} = 0$.

Boundary value problem

1ª riga: $\dot x = Ax + Bu$

2ª riga: $-\dot\mu = A^T\mu + Qx$

3ª riga: $0 = B^T\mu + Ru$ $\rightarrow$ $u = -R^{-1}B^T\mu$

Structure of the problem

$Q \geq 0 \quad R > 0$

$\lambda \mathcal{E} - \mathcal{A}$

$$\mathcal{E}\begin{bmatrix}\dot\mu\\ \dot x\\ \dot u\end{bmatrix} = \mathcal{A}\begin{bmatrix}\mu\\ x\\ u\end{bmatrix}, \quad \mathcal{A} = \begin{bmatrix}0 & A & B\\ A^T & Q & 0\\ B^T & 0 & R\end{bmatrix}, \mathcal{E} = \begin{bmatrix}0 & I & 0\\ -I & 0 & 0\\ 0 & 0 & 0\end{bmatrix}.$$

Pencils $\lambda\mathcal{E} - \mathcal{A}$ with $\mathcal{A} = \mathcal{A}^T$, $\mathcal{E} = -\mathcal{E}^T$ are called $\boxed{\text{even.}}$

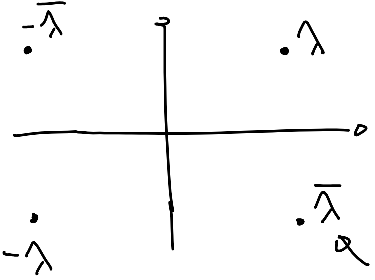
Eigenvalue pairing: if $(\lambda\mathcal{E} - \mathcal{A})v = 0$, then $v^T(-\overline{\lambda}\mathcal{E} - \mathcal{A}) = 0$, and $-\overline{\lambda}$ is an eigenvalue, too.

On a real problem, eigenvalues usually come in quadruples, $(\lambda, \overline{\lambda}, -\lambda, -\overline{\lambda})$. They may be degenerate if λ is real or pure imaginary.

$$(\lambda\mathcal{E} - \mathcal{A})v = 0 \Rightarrow 0 = v^*(\overline{\lambda}\mathcal{E}^* - \mathcal{A}^*) = v^*(-\overline{\lambda}\mathcal{E} - \mathcal{A})$$

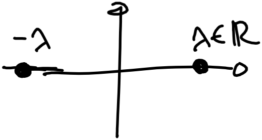
$\Rightarrow -\overline{\lambda}$ è un autovalore

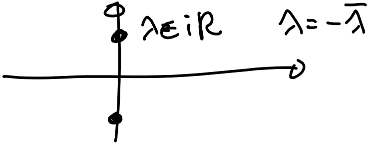
($f(x) = a_0 + a_1 x + a_2 x^2 + \dots$ è pari se $a_0, a_2, a_4 \dots$ reali, $a_1, a_3, a_5, \dots$ immag.

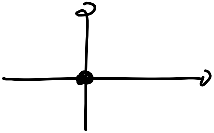


se $\mathcal{E}, A$ reali

(casi degeneri:







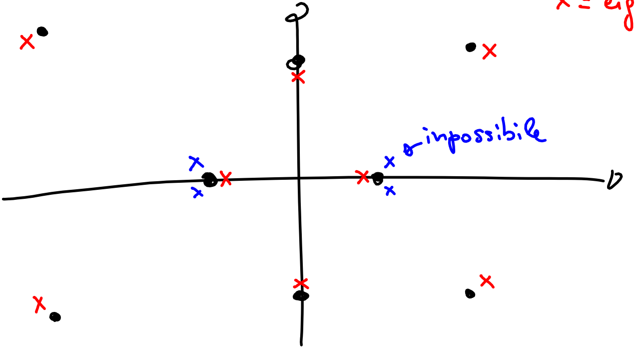
)

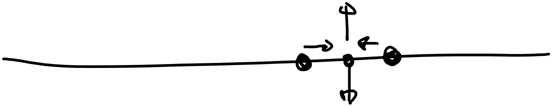
(Se ho una catena di Jordan $v_1, v_2, \dots v_k$ associata a λ, traspongo tutto e ne ottengo una associata a $-\bar{\lambda}$

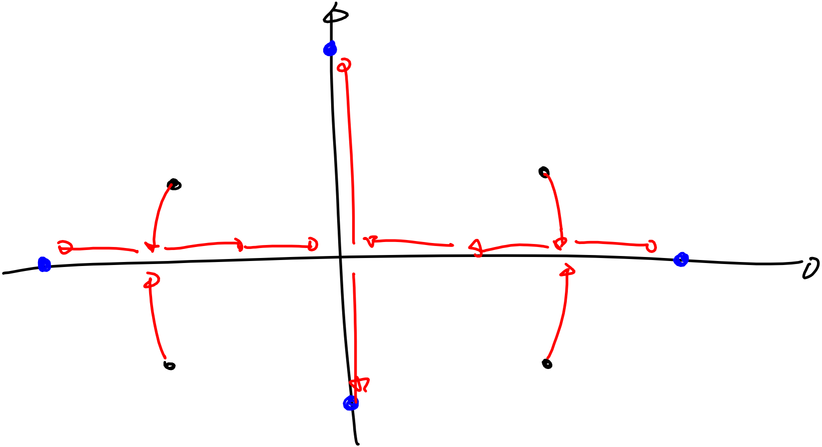
$\Rightarrow$ λ e $-\bar{\lambda}$ hanno la stessa molteplicità)

Cosa succede se perturbiamo A, E?
(mantenendo $\tilde{A} = \tilde{A}^T$, $\tilde{E} = \tilde{E}^T$)?

$\times = \mathrm{eig}(\tilde{A}, \tilde{E})$







The eigenvalues

If $R \succ 0$, row/column operations give

$$\lambda \mathcal{E} - \mathcal{A} \sim \lambda \begin{bmatrix} 0 & I & 0 \\ -I & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} -BR^{-1}B^T & A & 0 \\ A^T & Q & 0 \\ 0 & 0 & I \end{bmatrix}.$$

This shows that $\lambda \mathcal{E} - \mathcal{A}$ has m simple eigenvalues at ∞, plus $2n$ finite eigenvalues (with multiplicity): those of

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} & I \\ -I & \end{bmatrix}^{-1} \begin{bmatrix} -BR^{-1}B^T & A \\ A^T & Q \end{bmatrix} = \begin{bmatrix} -A^T & -Q \\ -BR^{-1}B^T & A \end{bmatrix}$$

$$\begin{bmatrix} -BR^{-1}B^T & A \\ A^T & Q \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^{-1}$$

$$\begin{bmatrix} -BR^{-1}B^T & A \\ A^T & Q \end{bmatrix} - \lambda \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & \lambda I - A & -B \\ -\lambda I - A^T & -Q & 0 \\ -B^T & 0 & -R \end{bmatrix} = \lambda \mathcal{E} - \mathcal{A}$$

Per trovare autovalori/vettori/forma di Kronecker, posso fare comb. lineari di righe/colonne,

$$\mathcal{A} - \lambda \mathcal{E} \sim S(\mathcal{A} - \lambda \mathcal{E})T$$

S, T invertibili

$$\begin{bmatrix} I & 0 & -BR^{-1} \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} (\lambda \mathcal{E} - \mathcal{A}) = \begin{bmatrix} BR^{-1}B^T & \lambda I - A & 0 \\ -\lambda I - A^T & -Q & 0 \\ -B^T & 0 & -R \end{bmatrix}$$

$$\begin{bmatrix} 1 & & -BR^{-1} \\ & 1 & \\ & & 1 \end{bmatrix} (\lambda \mathcal{E} - A) \begin{bmatrix} 1 & & \\ & 1 & \\ -R^{-1}B^T & & 1 \end{bmatrix} =$$

$$= \begin{array}{cccc} n & n & m & \\ \left[\begin{array}{cc|c} BR^{-1}B^T & \lambda I - A & 0 \\ -\lambda I - A^T & Q & 0 \\ \hline 0 & 0 & R \end{array}\right] & & & \begin{array}{c} n \\ n \\ m \end{array} \end{array}$$

$A \in \mathbb{R}^{n\times n}$ $B \in \mathbb{R}^{n\times m}$

$Q \in \mathbb{R}^{n\times n}$ $R \in \mathbb{R}^{m\times m}$

autovalori di questo pencil (e anche di $\lambda \mathcal{E} - A$):

- 1) Autoval. di $\begin{bmatrix} BR^{-1}B^T & \lambda I - A \\ -\lambda I - A^T & Q \end{bmatrix}$ ← $2n$ autoval. finiti
- 2) Autoval. di $0 \cdot \lambda + R$ ← m autoval. a ∞

1):

Autovalori di

$$\begin{bmatrix} BR^{-1}B^T & \lambda I - A \\ -\lambda I - A^T & -Q \end{bmatrix} = \lambda \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} - \begin{bmatrix} -BR^{-1}B^T & A \\ A^T & Q \end{bmatrix}$$

cioè autovel. della matrice

$$\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}^{-1} \begin{bmatrix} -BR^{-1}B^T & A \\ A^T & Q \end{bmatrix} \in \mathbb{R}^{2n \times 2n}$$

(2n, con molteplicità, tutti finiti)

Change of variables

The same idea, recast as a change of variables on the equations: μ, x, u solve

$$\rightsquigarrow \begin{bmatrix} 0 & I & 0 \\ -I & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\mu} \\ \dot{x} \\ \dot{u} \end{bmatrix} = \begin{bmatrix} 0 & A & B \\ A^T & Q & 0 \\ B^T & 0 & R \end{bmatrix} \begin{bmatrix} \mu \\ x \\ u \end{bmatrix}$$

iff $u = -R^{-1} B^T \mu$ and μ, x solve $\longrightarrow$ $0 = B^T\mu + Ru$

$u = -R^{-1}B^T\mu$

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \begin{bmatrix} & I \\ -I & \end{bmatrix} \overset{\begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}}{\begin{bmatrix} \dot{\mu} \\ \dot{x} \end{bmatrix}} = \begin{bmatrix} -BR^{-1}B^T & A \\ A^T & Q \end{bmatrix} \begin{bmatrix} \mu \\ x \end{bmatrix}$$

or

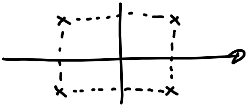
$$\begin{bmatrix} \dot{x} \\ \dot{\mu} \end{bmatrix} = \mathcal{H} \begin{bmatrix} x \\ \mu \end{bmatrix}, \quad \mathcal{H} = \begin{bmatrix} A & -G \\ -Q & -A^T \end{bmatrix}, \quad \boxed{G = BR^{-1}B^T.}$$

$x(0) = x_0$

$\lim_{t \to \infty} \begin{bmatrix} x \\ \mu \end{bmatrix} = 0$

$G = G^T \succcurlyeq 0 \qquad Q = Q^T \succcurlyeq 0$

Solving the reduced problem



Suppose that:

- $\mathcal{H}$ has n eigenvalues in the LHP and n in the RHP. (Recall: $\mathcal{H}$ has "even eigensymmetry").
- we find X such that $\underline{\begin{bmatrix} I \\ X \end{bmatrix}}$ spans the stable (eigenvalues $\in LHP$) invariant subspace of $\mathcal{H}$, i.e., $\mathcal{H} \begin{bmatrix} I \\ X \end{bmatrix} = \begin{bmatrix} I \\ X \end{bmatrix} \boxed{\mathcal{R}}$.

$v_i \exp(\lambda_i t)$
per (λ_i, v_i) coppia
autoval/vett con $\lambda_i \in LHP$

Then, the stable solutions of

$$\begin{bmatrix} \dot{x} \\ \dot{\mu} \end{bmatrix} = \mathcal{H} \begin{bmatrix} x \\ \mu \end{bmatrix}$$

are given by

$$\begin{bmatrix} x(t) \\ \mu(t) \end{bmatrix} = \boxed{\begin{bmatrix} I \\ X \end{bmatrix} \exp(\mathcal{R}t) \underset{\sim}{v}.}$$

The initial condition $x(0) = x_0$ gives $v = x_0$. Moreover, $\mu(t) = Xx(t)$, hence $u(t) = -R^{-1}B^T Xx(t)$.

Spazio stabile = span{autovettori e catene di Jordan relativi ad autovalori in LHP}

Stiamo supponendo che lo spazio stabile abbia dimensione n e che abbia una base del tipo $\begin{array}{c} n \\ \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \begin{matrix} n \\ n \end{matrix} \end{array}$ con U_1 invertibile, così posso ottenere un'altra base come $\begin{bmatrix} U_1 \\ U_2 \end{bmatrix} U_1^{-1} = \begin{bmatrix} I \\ U_2 U_1^{-1} \end{bmatrix} =$

$$= \begin{bmatrix} I \\ X \end{bmatrix}.$$

$[v^T \ 0] \cdot w = 0$ per ogni vettore dello spazio

Soluzione del BVP (sotto le due assunzioni):

$$(*) \quad \begin{bmatrix} x(t) \\ \mu(t) \end{bmatrix} = \begin{bmatrix} 1 \\ X \end{bmatrix} \exp(Rt)\, x_0 = \begin{matrix} \exp(Rt) \cdot x_0 \\ X \cdot x(t) \end{matrix}$$

$$(**) \quad u(t) = -R^{-1}B^T \mu(t) = -R^{-1}B^T X\, x(t)$$

1) calcola sottosp. stabile nella forma $\begin{bmatrix} 1 \\ X \end{bmatrix}$

2) $R = A - GX$

3) assembla soluzione $(*)$, $(**)$.

Algebraic Riccati equations

We have reduced the problem to $\mathcal{H}\begin{bmatrix} I \\ X \end{bmatrix} = \begin{bmatrix} I \\ X \end{bmatrix}\mathcal{R}$, or

$$\begin{array}{l} 1 \to \\ 2 \to \end{array} \begin{bmatrix} A & -G \\ -Q & -A^T \end{bmatrix}\begin{bmatrix} I \\ X \end{bmatrix} = \begin{bmatrix} I \\ X \end{bmatrix}\mathcal{R}$$

$$1\ \mathcal{R} = A - GX, \quad -Q - A^T X = XA - XGX.$$

$$2 \qquad \Updownarrow$$

$$\to \qquad A^T X + XA + Q - XGX = 0, \quad Q \succeq 0,\ G \succeq 0$$

is called algebraic Riccati equation. $\Lambda(A-GX) \leq LHP$

We look for a stabilizing solution, i.e., $\Lambda(\mathcal{R}) \subseteq LHP$.

(Note that $\Lambda(\mathcal{R}) \subset \Lambda(\mathcal{H})$.)

Next goal: show that we can do what we claimed in the previous slide.

Solvability conditions

Solutions of (ARE) $\iff$ n-dimensional invariant subspaces of $\mathcal{H}$ with invertible top block.

If $\mathcal{H}$ has distinct eigenvalues, there are at most $\binom{2n}{n}$ solutions (choose n eigenvalues out of the $2n\dots$)

Does it have a (unique) stabilizing solution? $\mathcal{H}$ Must have (exactly) n eigenvalues in the LHP, and the associated invariant subspace must be expressible as $\operatorname{Im}\begin{bmatrix} I \\ X \end{bmatrix}$.

Hamiltonian matrices

$$\boxed{\mathcal{H} = \begin{bmatrix} A & -G \\ -Q & -A^* \end{bmatrix}}, \quad Q = Q^*,\ G = G^*$$

is a Hamiltonian matrix, i.e., it satisfies $\boxed{J\mathcal{H} = -\mathcal{H}^* J}$, where
$J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$.
(Skew-self-adjoint with respect to the antisymmetric scalar product defined by J.)

If $\underline{\mathcal{H}v = \lambda v}$, then $(v^* J)\mathcal{H} = (-\overline{\lambda})(v^* J)$: eigenvalues have 'even symmetry', and the right eigenvector relative to λ is related to the left one relative to $-\overline{\lambda}$.

A similar relation can be proved for Jordan chains: λ and $-\overline{\lambda}$ have Jordan chains of the same size.

Prodotto scalare: $\langle u, v \rangle_J = u^* J v = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}^* \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$
(indefinito)

$$= u_1^* v_2 - u_2^* v_1$$

Matrice autoaggiunta se $\langle u, Mv \rangle_J = \langle Mu, v \rangle_J$

$$\Leftrightarrow \quad u^* J M v = u^* M^* J v \quad \Leftrightarrow \quad JM = M^* J$$

Matrice skew-self-adjoint $\Leftrightarrow JM = -M^* J$

$\mathcal{H}$ è skew-self adjoint:

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} A & -G \\ -Q & -A^T \end{bmatrix} = \begin{bmatrix} -Q & -A^T \\ -A & G \end{bmatrix}$$

$$-\begin{bmatrix} A^T & -Q \\ -G & -A \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = -\begin{bmatrix} Q & A^T \\ A & -G \end{bmatrix}$$

(tutte le matrici della forma $\begin{bmatrix} A & -G \\ -Q & -A^T \end{bmatrix}$, con $G = G^T$, $Q = Q^T$

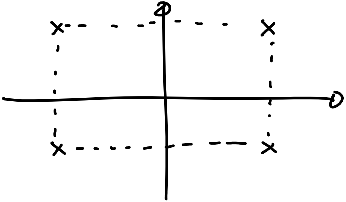
sono skew-self-adjoint rispetto a quel prod. scalare)

$$J\mathcal{H} = -\mathcal{H}^* J \qquad \mathcal{H}v = \lambda v$$

$$v^* \bar{\lambda} = v^* \mathcal{H}^* = v^*(-J\mathcal{H}J^{-1}) \Leftrightarrow (-\bar{\lambda})v^* J = v^* J \mathcal{H}$$

Lo stesso conto vale per le catene di Jordan

$\Rightarrow$ gli autovalori di $\mathcal{H}$ hanno simmetria rispetto all'asse immaginario



(Nota che autoval. di $\mathcal{H}$ $\Leftrightarrow$ autoval. di $\lambda J - \begin{bmatrix} -BR^{-1}B^T & A \\ A^T & Q \end{bmatrix}$, che è una pencil pari (antisimmetrica - simmetrica))

$\Rightarrow$ Per dire che $\mathcal{H}$ ha n autoval. in LHP, mi basta dire che non ha autoval. immaginari.

Solvability conditions

Theorem

Assume $Q \succ 0$, and (A, B) stabilizable. Then, $\mathcal{H}$ has no eigenvalues with $\operatorname{Re}\lambda = 0$.

($Q \succ 0$ can be weakened to $Q \succeq 0$ and (A^T, Q^T) stabilizable.)

Proof (sketch)

Suppose instead $\mathcal{H}\left[\begin{smallmatrix} z_1 \\ z_2 \end{smallmatrix}\right] = \imath\omega\left[\begin{smallmatrix} z_1 \\ z_2 \end{smallmatrix}\right]$; from $0 = \operatorname{Re}\left[\begin{smallmatrix} z_2^* & z_1^* \end{smallmatrix}\right]\left[\begin{smallmatrix} A & -G \\ -Q & -A^* \end{smallmatrix}\right]\left[\begin{smallmatrix} z_1 \\ z_2 \end{smallmatrix}\right] = z_2^* G z_2 + z_1^* Q z_1$ follows that $z_1 = 0$, $z_2^* B = 0$. But the latter together with $z_2^* A = -\imath\omega z_2^*$ contradicts stabilizability.

Hence, $\mathcal{H}$ has n eigenvalues in the LHP and n associated ones in the RHP: it has exactly one stabilizing subspace.

- Lancaster, Rodman, "Algebraic Riccati equations"
- Bini, Iannazzo, Meini, "Numerical solution of AREs"
- Benner, Mehrmann "Kontrolltheorie" (tedesco!)

Form of the invariant subspace

We know now that there exist $U_1, U_2 \in \mathbb{R}^{n\times n}$ such that $\begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$ spans the stable invariant subspace.
Moreover, $\begin{bmatrix} U_1^* & U_2^* \end{bmatrix} J = \begin{bmatrix} U_2^* & -U_1^* \end{bmatrix}$ spans the left anti-stable invariant subspace.

Left and right invariant subspaces relative to disjoint eigenvectors are orthogonal $\implies$

$$0 = \begin{bmatrix} U_2^* & -U_1^* \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = U_2^* U_1 - U_1^* U_2.$$

We'd like to show that U_1 is invertible. Then (up to changing basis in $\begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$) we can take $U_1 = I$, $U_2 = X = X^*$.

$\text{span}\begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$ = sottosp. inv. stabile $\quad U_1, U_2 \in \mathbb{R}^{n \times n}$

(U₁ invertibile)

$$\text{span}\, J \cdot \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = \text{span}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = \text{span}\begin{bmatrix} U_2 \\ -U_1 \end{bmatrix} =$$

sottospazio instabile sinistro (cioè, autovett. e catene di Jordan sinistre relative a autovettori $\in$ RHP)

autovett. sinistri e destri relativi a sottosp. diversi sono ortogonali:

$$[U_1\ U_2]^* J \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = 0$$

$$\Longleftrightarrow \quad U_1^* U_2 - U_2^* U_1 = 0$$

Questo vale per qualunque base $\begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$ del sottosp. invariante stabile, quindi anche per

$$\begin{bmatrix} 1 \\ X \end{bmatrix} = \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} \quad \Rightarrow X - X^* = 0$$

Nonsingularity of U_1

Suppose (A, B) stabilizable, $Q \succeq 0$, $G \succeq 0$. Then U_1 is invertible.

We'd like to show that U_1 is nonsingular. Suppose otherwise $U_1 v = 0$, $U_2 v \neq 0$. Then,

$$-v^* U_2^* G U_2 v = \begin{bmatrix} v^* U_2^* & 0 \end{bmatrix} \mathcal{H} \begin{bmatrix} 0 \\ U_2 v \end{bmatrix} = v^* \begin{bmatrix} U_2^* & U_1^* \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \mathcal{R} = 0.$$

implies $BU_2 v = 0$.
The first block row gives $U_1 \mathcal{R} v = 0 \implies U_1$ is $\mathcal{R}$-invariant and there are x, $\lambda \in LHP$ such that $U_1 x = 0$, $\mathcal{R} x = \lambda x$. Now the second block rows gives $-A^T U_2 x = \lambda U_2 x$. This (together with $BU_2 x = 0$ from above) contradicts stabilizability.

How to solve Riccati equations

(Schur)

- Newton's method.
- Invariant subspace via unstructured methods (QR).
- Invariant subspace via 'semi-structured' methods (Laub trick).
- Invariant subspace via structured methods (URV).
- Doubling / Sign iteration.