Sign-like methods for CAREs

Matrix sign iteration

$$\boxed{X_{k+1} = \frac{1}{2}(X_k + X_k^{-1}), \quad X_0 = \mathcal{H}.}$$

It is not difficult to see that X_k is Hamiltonian at each step (i.e., $JX_k = -X_k^* J$). Just show that

- If M is Hamiltonian, then M^{-1} is Hamiltonian, too.
- If M_1, M_2 are Hamiltonian, then $M_1 + M_2$ is Hamiltonian, too.

(Guiding idea: Hamiltonian matrices are ‘like antisymmetric ones’: properties that you expect for antisymmetric matrices will often hold for Hamiltonian, too.)

Lemma: Se H_1, H_2 Hamiltoniane (i.e. $\underbrace{H_i^* J = -J H_i}_{(\ast)}$),

allora

- $H_1 + H_2$ è Hamiltoniana

  (basta sommare $(\ast)$ per $i = 1, 2$)

- H_1^{-1} è Hamiltoniana:

$$H_1^* J = -J H_1 \rightarrow J H_1^{-1} = -H_1^{-*} J$$

$$\underline{H^* J = -JH} \qquad J^* = -J \qquad J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$$

$$(JH)^* = -\left(H^* J\right) = H^* J^*$$

$$\left[\begin{array}{c|c} A & -Q \\ \hline -G & -A^T \end{array}\right] \qquad JH = \begin{bmatrix} -G & -A^T \\ -A & Q \end{bmatrix}$$

$$X_{k+1} = \frac{1}{2}\left(X_k + X_k^{-1}\right) \qquad Z_k := J X_k$$

$$\boxed{Z_{k+1}} = J X_{k+1} = J \cdot \frac{1}{2}\left(X_k + X_k^{-1}\right) = \frac{1}{2}\left(J X_k + J X_k^{-1}\right) =$$

$$= \boxed{\frac{1}{2}\left(Z_k + J Z_k^{-1} J\right)}$$

(per calcolare Z_k^{-1} posso usare un metodo per matrici simmetriche, ad es. LDL^T, che costa la metà di un metodo per matrici generiche

Structure-preserving sign iteration

In machine arithmetic, the X_k won't be exactly Hamiltonian — unless we modify our algorithm to ensure that they are.

Recall: X is Hamiltonian iff $Z = JX$ is symmetric.
Rewrite the iteration in terms of $\boxed{Z_k := JX_k:}$

$$Z_{k+1} = \frac{1}{2}(Z_k + J\underset{\sim}{Z_k^{-1}}J), \quad Z_0 = J\mathcal{H}.$$

Will preserve symmetry exactly (assuming the method we use for inversion does).

We can incorporate <u>scaling.</u>

It is in some sense 'working on even pencils': given an even pencil $\lambda J - Z_k$, construct $\lambda J - Z_{k+1}$ (will see more of this idea in the following).

$\underline{J\lambda - Z_k}$ ha gli stessi autovalori

di $J^{-1}Z_k = X_k$

Problema del metodo: serve invertire X_k o Z_k a ogni passo. Se Z_0, o X_0, è quasi singolare, allora posso fare errori al primo passo e non recupero più.

Towards doubling

$$\begin{tikzcd} Y_k \arrow[r] \arrow[d, leftrightarrow] & -Y_k^2 \arrow[d, leftrightarrow] \\ X_k \arrow[r] & \frac{1}{2}(X_k + X_k^{-1}) \end{tikzcd}$$

Recall: in the ~~doubling~~ sign iteration, if we set $\underline{Y_k = (I - X_k)^{-1}(I + X_k)}$, then $\underline{Y_{k+1} = -Y_k^2}$.

In an ideal world without rounding errors, we could compute $\underline{Y_0, Y_1, Y_2}, \dots$, and then get the stable invariant subspace as $\ker Y_\infty$.

We can do something similar, if we work in a suitable format.

X_0 ha n autovalori in LHP, n in RHP

Y_0 ha n autoval. $\lambda_1, \lambda_2 \dots \lambda_n$ con $|\lambda_i| < 1$, n con $|\lambda_i| > 1$

$$Y_k = (Y_0)^{2^k} \text{ ha autoval. } \lambda_i^{2^k} \begin{cases} n \text{ tendono a } 0 \\ n \text{ tendono a } \infty \end{cases}$$

Vogliamo avere ad ogni passo

$$Y_k = \begin{bmatrix} I & G_k \\ O & F_k \end{bmatrix}^{-1} \begin{bmatrix} E_k & O \\ H_k & I \end{bmatrix}$$

per opportune matrici E_k, F_k, G_k, H_k

$$Y_0 = (I - \mathcal{H})^{-1}(I + \mathcal{H})$$

Trucco: cerco una matrice $M \in \mathbb{C}^{2n \times 2n}$ invertibile tale che

$$M \Big[\underbrace{I-\mathcal{H}}_{2n} \;\vdots\; \underbrace{I+\mathcal{H}}_{2n}\Big]_{2n} = \left[\begin{array}{cc:cc} I & G_0 & E_0 & O \\ O & F_0 & H_0 & I \end{array}\right]^{2n}_{\underbrace{\hphantom{xxx}}_{2n} \; \underbrace{\hphantom{xxx}}_{2n}}.$$

Se vale questa identità, allora

$$M(I-\mathcal{H}) = \begin{bmatrix} 1 & G_0 \\ 0 & F_0 \end{bmatrix} \qquad \Rightarrow$$

$$M(I+\mathcal{H}) = \begin{bmatrix} E & 0 \\ H_0 & 1 \end{bmatrix}$$

$$(I-\mathcal{H})^{-1}\cancel{M^{-1}M}(I+\mathcal{H}) = \begin{bmatrix} I & G_0 \\ 0 & F_0 \end{bmatrix}^{-1}\begin{bmatrix} E_0 & 0 \\ H_0 & I \end{bmatrix}$$

Come trovo M che fa questo lavoro?

$$M\left[I-\mathcal{H} \;\vdots\; I+\mathcal{H}\right] = M\left[\begin{array}{cc:cc} I-A & G & I+A & -G \\ \underset{\sim}{Q} & I+A^T & -Q & \underset{\sim}{I-A^T} \end{array}\right]$$

$$\overset{!}{=}\left[\begin{array}{cc|cc} I & G_0 & E_0 & 0 \\ \underset{\sim}{0} & F_0 & H_0 & \underset{\sim}{I} \end{array}\right]$$

vale se e solo se

$$M\begin{bmatrix} I-A & -G \\ Q & I-A^T \end{bmatrix} = I \quad e \quad M\begin{bmatrix} G & I+A \\ I+A^T & -Q \end{bmatrix} = \begin{bmatrix} G_0 & E_0 \\ F_0 & H_0 \end{bmatrix}$$

(1^a, 4^a colonne) (2^a, 3^a colonne)

$$\begin{bmatrix} E_0 & G_0 \\ H_0 & F_0 \end{bmatrix} = \begin{bmatrix} I-A & -G \\ Q & I-A^* \end{bmatrix}^{-1} \begin{bmatrix} I+A & G \\ -Q & I+A^* \end{bmatrix} \tag{*}$$

Struttura di queste matrici:

$(I-\mathcal{H})^{-1}(I+\mathcal{H})$ è simplettica:

[parallelo del fatto che se B è antisimmetrica, allora $(I-B)^{-1}(I+B)$ è ortogonale]

[perché B è del tipo $Q\Lambda Q^T$ con Λ immag. puri, e $\frac{1+x}{1-x}$ manda $i\mathbb{R}$ nel cerchio unitario]

Lemma: se $\mathcal{H}$ Hamiltoniana, $\mathcal{H}^* J = - J \mathcal{H}$, allora $S = (1-\mathcal{H})^{-1}(1+\mathcal{H})$ è simplettica, cioè

$$S^* J S = J$$

$$(1-\mathcal{H})^{-*}(1+\mathcal{H}^*) J (1+\mathcal{H})(1-\mathcal{H})^{-1} \overset{?}{=} J$$

$$(1+\mathcal{H})^* J (1+\mathcal{H}) \overset{?}{=} (1-\mathcal{H})^* J (1-\mathcal{H})$$

$$\cancel{J} + \mathcal{H}^* J + J\mathcal{H} + \cancel{\mathcal{H}^* J \mathcal{H}} \overset{?}{=} \cancel{J} - \mathcal{H}^* J - J\mathcal{H} + \cancel{\mathcal{H}^* J \mathcal{H}}$$

$$2\mathcal{H}^* J \overset{?}{=} -2J\mathcal{H}$$

✓ perché equivale a $\mathcal{H}$ Hamiltoniana

(Volendo, è un risultato su pencil:

Def: una pencil $\lambda \mathcal{E} - A$ è simplettica se $\mathcal{E} J \mathcal{E}^* = A J A^*$

$\left(\Leftrightarrow \quad (\mathcal{E}^{-1}A)^* J (\mathcal{E}^{-1}A) = J \right) \quad)$

Quando è che $\begin{bmatrix} 1 & G_0 \\ 0 & F_0 \end{bmatrix}^{-1} \begin{bmatrix} E_0 & 0 \\ H_0 & 1 \end{bmatrix}$ è simplettica?

Quando

$$\begin{bmatrix} E_0 & 0 \\ H_0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} E_0 & 0 \\ H_0 & 1 \end{bmatrix}^* = \begin{bmatrix} 1 & G_0 \\ 0 & F_0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & G_0 \\ 0 & F_0 \end{bmatrix}^*$$

$$\begin{bmatrix} 0 & E_0 \\ -1 & H_0 \end{bmatrix} \begin{bmatrix} E_0^* & H_0^* \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -G_0 & 1 \\ -F_0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ G_0^* & F_0^* \end{bmatrix}$$

$$\begin{bmatrix} 0 & E_0 \\ -E_0^* & -H_0^* + H_0 \end{bmatrix} = \begin{bmatrix} -G_0 + G_0^* & F_0^* \\ -F_0 & 0 \end{bmatrix}$$

$\Leftrightarrow \; G_0 = G_0^*, \; H_0 = H_0^*, \; E_0 = F_0^*$

$$(\mathcal{E}^{-1}A)^{*}J(\mathcal{E}^{-1}A)=J$$

$$\updownarrow$$

$$A^{*}\mathcal{E}^{-*}J\mathcal{E}^{-1}A=J$$

$$\updownarrow$$

$$\mathcal{E}^{-*}J\mathcal{E}^{-1}=A^{-*}JA^{-1}$$

$$\updownarrow$$

$$\boxed{\mathcal{E}J\mathcal{E}^{*}=AJA^{*}}$$

Standard Symplectic Form

Goal: write $Y_0 = (I - \mathcal{H})^{-1}(I + \mathcal{H})$ as

$$\boxed{Y_0 = \begin{bmatrix} I & -G_0 \\ 0 & F_0 \end{bmatrix}^{-1} \begin{bmatrix} E_0 & 0 \\ -H_0 & I \end{bmatrix}.}$$

Can be rewritten as: find M such that

$$M \begin{bmatrix} (I - \mathcal{H}) & (I + \mathcal{H}) \end{bmatrix} = \begin{bmatrix} I & G_0 & E_0 & 0 \\ 0 & F_0 & H_0 & I \end{bmatrix}.$$

Solution: M is the inverse of block columns $(1, 4)$.

Structural properties: if $\mathcal{H}$ is Hamiltonian, Y_0 is symplectic. If Y_0 is symplectic, $\boxed{E_0 = F_0^*, G_0 = G_0^*, H_0 = H_0^*.}$
Moreover, if $G \succeq 0$, $H \succeq 0$, then $\boxed{G_0 \succeq 0, H_0 \preceq 0.}$

(All these manipulations can be reformulated as left-multiplication of the pencil $(I - \mathcal{H}, I + \mathcal{H})$.)

$$Y_0 \to Y_1 = -Y_0^2 \qquad Y_k \to Y_{k+1} = -Y_k^2$$

$$Y_{k+1} = \underbrace{\begin{bmatrix} 1 & G_{k+1} \\ 0 & f_{k+1} \end{bmatrix}^{-1}}_{M_{k+1}^{-1}} \underbrace{\begin{bmatrix} E_{k+1} & 0 \\ H_{k+1} & 1 \end{bmatrix}}_{N_{k+1}} = -\Bigg(\underbrace{\begin{bmatrix} 1 & G_k \\ 0 & f_k \end{bmatrix}^{-1}}_{M_k^{-1}} \underbrace{\begin{bmatrix} E_k & 0 \\ H_k & 1 \end{bmatrix}}_{N_k} \Bigg)^2$$

Visto come mappa

$$(E_k, f_k, G_k, H_k) \to (E_{k+1}, F_{k+1}, G_{k+1}, H_{k+1})$$

Abbiamo M_k, N_k, vogliamo fattorizzare

$$M_{k+1}^{-1} N_{k+1} = -\left(M_k^{-1} N_k\right)^2 = -M_k^{-1} \underleftrightarrow{N_k M_k^{-1}} N_k$$

Se troviamo $\hat{M}_k, \hat{N}_k$ tali che $N_k M_k^{-1} = \hat{M}_k^{-1} \hat{N}_k$, Abbiamo

$$-M_k^{-1}\underbrace{N_k M_k^{-1}}_{}N_k = -M_k^{-1}\underbrace{\hat{M}_k^{-1}\hat{N}_k}_{}N_k =$$

$$= \underbrace{\left(-\hat{M}_k M_k\right)^{-1}}_{M_{k+1}^{-1}}\underbrace{\left(\hat{N}_k N_k\right)}_{N_{k+1}}.$$

$$N_k M_k^{-1} = \hat{M}_k^{-1}\hat{N}_k \iff \hat{M}_k N_k = \hat{N}_k M_k$$

$$\iff \begin{bmatrix} -\hat{M}_k & \hat{N}_k \end{bmatrix}\begin{bmatrix} N_k \\ M_k \end{bmatrix} = 0$$

$$\Leftrightarrow \quad \left[\begin{array}{cc:cc} 1 & \hat{G}_k & \hat{E}_k & 0 \\ 0 & \hat{F}_k & \hat{H}_k & 1 \end{array}\right] \begin{bmatrix} E_k & 0 \\ H_k & 1 \\ 1 & G_k \\ 0 & F_k \end{bmatrix} = 0$$

$$\Leftrightarrow \quad \left[\begin{array}{cc:cc} 1 & 0 & \hat{E}_k & \hat{G}_k \\ 0 & 1 & \hat{H}_k & \hat{F}_k \end{array}\right] \left[\begin{array}{cc} E_k & 0 \\ 0 & F_k \\ \hdashline 1 & G_k \\ H_k & 1 \end{array}\right] = 0$$

$4n \times 2n$, ci aspettiamo che il suo kernel sx abbia dimensione $2n$, quindi di poter trovare una matrice con $2n$ righe lin. indipendenti.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} E_k & 0 \\ 0 & F_k \end{bmatrix} + \begin{bmatrix} \hat{E}_k & \hat{G}_k \\ \hat{H}_k & \hat{F}_k \end{bmatrix} \begin{bmatrix} 1 & G_k \\ H_k & 1 \end{bmatrix} = 0$$

$$\begin{pmatrix} \hat{E}_k & \hat{G}_k \\ \hat{H}_k & \hat{F}_k \end{pmatrix} = -\begin{bmatrix} E_k & 0 \\ 0 & F_k \end{bmatrix} \begin{bmatrix} 1 & G_k \\ H_k & 1 \end{bmatrix}^{-1} =$$

$$= -\begin{bmatrix} E_k & 0 \\ 0 & F_k \end{bmatrix} \cdot \begin{bmatrix} (1-G_kH_k)^{-1} & -G_k(1-H_kG_k)^{-1} \\ -H_k(1-G_kH_k)^{-1} & (1-H_kG_k)^{-1} \end{bmatrix}$$

$$= \begin{bmatrix} -E_k(1-G_kH_k)^{-1} & E_kG_k(1-H_kG_k)^{-1} \\ F_kH_k(1-G_kH_k)^{-1} & -F_k(1-H_kG_k)^{-1} \end{bmatrix}.$$

$$\begin{bmatrix} 1 & G_k \\ H_k & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -G_k \\ -H_k & 1 \end{bmatrix} = \begin{bmatrix} 1-G_kH_k & 0 \\ 0 & 1-H_kG_k \end{bmatrix}$$

$$M_{k+1} = \left(-\hat{M}_k\right) M_k = \begin{bmatrix} I & \hat{G}_k \\ 0 & \hat{F}_k \end{bmatrix} \begin{bmatrix} 1 & G_k \\ 0 & F_k \end{bmatrix} =$$

$$= \begin{bmatrix} I & E_k G_k \left(1 - H_k G_k\right)^{-1} \\ 0 & -F_k \left(1 - H_k G_k\right)^{-1} \end{bmatrix} \begin{bmatrix} I & G_k \\ 0 & F_k \end{bmatrix} = \begin{bmatrix} I & \overset{G_{k+1}}{\overset{\parallel}{\boxed{G_k + E_k G_k \left(1 - H_k G_k\right)^{-1} F_k}}} \\ 0 & \underset{F_{k+1}}{\underset{\parallel}{\boxed{-F_k \left(1 - H_k G_k\right)^{-1} F_k}}} \end{bmatrix}$$

$$N_{k+1} = \hat{N}_k N_k = \begin{bmatrix} \hat{E}_k & 0 \\ \hat{H}_k & 1 \end{bmatrix} \begin{bmatrix} E_k & 0 \\ H_k & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} -E_k \left(1 - G_k H_k\right)^{-1} & 0 \\ F_k H_k \left(1 - G_k H_k\right)^{-1} & 1 \end{bmatrix} \begin{bmatrix} E_k & 0 \\ H_k & 1 \end{bmatrix} = \begin{bmatrix} \overset{E_{k+1}}{\overset{\parallel}{\boxed{-E_k \left(1 - G_k H_k\right)^{-1} E_k}}} & 0 \\ \underset{H_{k+1}}{\underset{\parallel}{\boxed{H_k + F_k H_k \left(1 - G_k H_k\right)^{-1} E_k}}} & I \end{bmatrix}$$

Structured doubling algorithm

$$\begin{bmatrix} E_0 & G_0 \\ H_0 & F_0 \end{bmatrix} = \begin{bmatrix} I-A & -G \\ Q & I-A^T \end{bmatrix}^{-1} \begin{bmatrix} I+A & G \\ Q & I+A^T \end{bmatrix}.$$

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for $k = 0, 1, 2, \dots$
    $E_{k+1} = -E_k (I - G_k H_k)^{-1} E_k$
    $F_{k+1} = -F_k (I - H_k G_k)^{-1} F_k$
    $G_{k+1} = G_k + E_k G_k (I - H_k G_k)^{-1} F_k$
    $H_{k+1} = H_k + F_k H_k (I - G_k H_k)^{-1} E_k$
end
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scambiare

$E_k \leftrightarrow F_k$

$G_k \leftrightarrow H_k$

lascia invariata

Per come le abbiamo derivate, X_k dell'it. segno soddisfa

$$(I - X_k)^{-1}(I + X_k) = \begin{bmatrix} I & G_k \\ 0 & F_k \end{bmatrix}^{-1} \begin{bmatrix} E_k & 0 \\ H_k & I \end{bmatrix}$$

Struttura: ad ogni passo abbiamo:

(1) $E_k = F_k^*$ (2) $G_k = G_k^*$ (3) $H_k = H_k^*$ (simpletticità)

e anche $0 \preceq G_0 \preceq G_1 \preceq G_2 \preceq \dots$ (4)

$0 \succeq H_0 \succeq H_1 \succeq H_2 \succeq \dots$ (5)

<u>dim</u>: Per induzione

(1): Facile, basta fare $F_{k+1}^* = F_k^* (1 - H_k G_k)^{-*} F_k^* =$

$$= E_k (1 - G_k H_k)^{-1} E_k$$

(2) Ci basta dire che $G_k (1 - H_k G_k)^{-1}$ è simmetrica

Se posso espandere la serie, vale

$$G_k (1 - H_k G_k)^{-1} = G_k + G_k H_k G_k + G_k H_k G_k H_k G_k + \dots$$

$$= (1 - G_k H_k)^{-1} G_k$$

Anche se non posso espandere la serie, vale $G_K(1-H_KG_K)^{-1}=(1-G_KH_K)^{-1}G_K$

(è un'identità: $(1-G_KH_K)G_K=G_K(1-H_KG_K)$)

Questo mi dice anche che

$$\left(G_K(1-H_KG_K)^{-1}\right)^*=(1-G_KH_K)^{-1}G_K=G_K(1-H_KG_K)^{-1}$$

è simmetrica

(3) è uguale a (2)

(4) Mi basta dire che $G_K(1-H_KG_K)^{-1}\geqslant 0$

Posso sempre scrivere $G_K=R_KR_K^*$, e ho

$$G_K(1-H_KG_K)^{-1}=R_K(I-R_K^*H_KR_K)^{-1}R_K^*.$$

Se $H_k \preceq 0$, allora

$$R_k \left(I - \underbrace{R_k^* H_k R_k}_{\preceq 0} \right)^{-1} R_k^* \succeq 0$$

(5) uguale a (4).

[Notate che le inversioni che servono si riescono a fare, perché $R_k^* H_k R_k \succeq 0$, quindi $I + R_k^* H_k R_k \succ 0$, e similmente anche $I - G_k H_k$

$I - H_k G_k$

non sono mai singolari (. .

($G_k H_k$ ha gli stessi autoval. di $H_k G_k$
e di $R_k^* H_k R_k$, che sono tutti ≤ 0.)

Costo: l'eq. per F_{k+1} non serve (è il trasposto di E_{k+1}). Le inversioni si riducono a una sola, perché $I - H_k G_k = (I - G_k H_k)^*$, e anzi con un po' di attenzione ci si riduce a invertire una matrice simmetrica.

Costo ~ comparabile con l'iteraz. segno
(in cui c'è da invertire una matrice $2n \times 2n$)

Doubling algorithm

Plan Given $Y_k = \begin{bmatrix} I & G_k \\ 0 & E_k^* \end{bmatrix}^{-1} \begin{bmatrix} E_k & 0 \\ H_k & I \end{bmatrix}$, compute $Y_{k+1} = -Y_k^2 = \begin{bmatrix} I & G_{k+1} \\ 0 & E_{k+1}^* \end{bmatrix}^{-1} \begin{bmatrix} E_{k+1} & 0 \\ H_{k+1} & I \end{bmatrix}$.

Similar to the 'inverse-free sign method' described earlier.

If $Y_k = \mathcal{M}_k^{-1}\mathcal{N}_k$, then $-Y_k^2 = -\mathcal{M}_k^{-1}\mathcal{N}_k\mathcal{M}_k^{-1}\mathcal{N}_k = \mathcal{M}_k^{-1}\widehat{\mathcal{M}}_k^{-1}\widehat{\mathcal{N}}_k\mathcal{N}_k = (\widehat{\mathcal{M}}_k\mathcal{M}_k)^{-1}(\widehat{\mathcal{N}}_k\mathcal{N}_k)$, where $\widehat{\mathcal{M}}_k, \widehat{\mathcal{N}}_k$ satisfy $\widehat{\mathcal{M}}_k^{-1}\widehat{\mathcal{N}}_k = -\mathcal{N}_k\mathcal{M}_k^{-1}$, i.e.,

$$\begin{bmatrix} \widehat{\mathcal{M}}_k & \widehat{\mathcal{N}}_k \end{bmatrix} \begin{bmatrix} \mathcal{N}_k \\ \mathcal{M}_k \end{bmatrix} = 0.$$

Doubling: inversion trick

$$\begin{bmatrix} I & \widehat{G}_k & \widehat{E}_k & 0 \\ 0 & \widehat{F}_k & \widehat{H}_k & I \end{bmatrix} \begin{bmatrix} E_k & 0 \\ H_k & I \\ I & G_k \\ 0 & E_k^* \end{bmatrix} = 0$$

holds if

$$\begin{aligned} \begin{bmatrix} \widehat{G}_k & \widehat{E}_k \\ \widehat{F}_k & \widehat{H}_k \end{bmatrix} &= -\begin{bmatrix} E_k & 0 \\ 0 & E_k^* \end{bmatrix} \begin{bmatrix} H_k & I \\ I & G_k \end{bmatrix}^{-1} \\ &= \begin{bmatrix} E_k & 0 \\ 0 & E_k^* \end{bmatrix} \begin{bmatrix} G_k(I - H_k G_k)^{-1} & -(I - G_k H_k)^{-1} \\ -(I - H_k G_k)^{-1} & H_k(I - G_k H_k)^{-1} \end{bmatrix}. \end{aligned}$$

Doubling: the formulas

Putting everything together,

$$\begin{aligned}\begin{bmatrix} E_{k+1} & 0 \\ H_{k+1} & I \end{bmatrix} &= \begin{bmatrix} -E_k(I - G_k H_k)^{-1} & 0 \\ E_k^* H_k (I - G_k H_k)^{-1} & I \end{bmatrix} \begin{bmatrix} E_k & 0 \\ H_k & I \end{bmatrix} \\ &= \begin{bmatrix} -E_k(I - G_k H_k)^{-1} E_k & 0 \\ H_k + E_k^* H_k (I - G_k H_k)^{-1} E_k & I \end{bmatrix}\end{aligned}$$

and an analogous computation gives E_{k+1}^*, G_{k+1}:

Structured doubling algorithm

$$\begin{aligned} E_{k+1} &= -E_k(I - G_k H_k)^{-1} E_k, \\ G_{k+1} &= G_k + E_k G_k (I - H_k G_k)^{-1} E_k^*, \\ H_{k+1} &= H_k + E_k^* H_k (I - G_k H_k)^{-1} E_k. \end{aligned}$$

SDA: details

Note that (even when the middle term does not converge)

$$G_k(I-H_kG_k)^{-1} = G_k+G_kH_kG_k+G_kH_kG_kH_kG_k+\cdots = (I-G_kH_k)^{-1}G_k,$$

and this matrix is symmetric. If $G_k = B_kB_k^*$, then it can also be rewritten as $B_k(I - B_k^*H_kB_k)^{-1}B_k^*$ (inverting a symmetric matrix).

Monotonicity If $H_k \preceq 0$ then $G_k(I - H_kG_k)^{-1} \succeq 0$. Hence, $0 \preceq G_0 \preceq G_1 \preceq \dots$, and $0 \succeq H_0 \succeq H_1 \succeq H_2 \succeq \dots$

Cost As much as a $2n \times 2n$ inversion $M^{-1}N$, if you put everything together. Unlike the sign algorithm, we have a bound $\sigma_{\min}(I - H_kG_k) \geq 1$.

SDA: the dual equation

To analyze convergence, we need to introduce another matrix. Let Y be the matrix such that

$$\mathcal{H}\begin{bmatrix}-Y\\ I\end{bmatrix} = \begin{bmatrix}A & -G\\ -Q & -A^*\end{bmatrix}\begin{bmatrix}-Y\\ I\end{bmatrix} = \begin{bmatrix}-Y\\ I\end{bmatrix}\widehat{\mathcal{R}}$$

is the anti-stable invariant subspace of $\mathcal{H}$, i.e., $\Lambda(\widehat{\mathcal{R}}) \subset RHP$.

Why does it exist? Because $\begin{bmatrix}I\\ Y\end{bmatrix}$ spans the stable subspace of $\mathcal{H}^* = -J\mathcal{H}J$, and we can repeat our arguments on it (in the version with (A, B) and (A^T, C^T) controllable)

SDA: convergence (intuitively)

Intuitive view E_k approximately squared at each time. Hence

$$\mathcal{H}_k = \begin{bmatrix} I & G_k \\ 0 & E_k^* \end{bmatrix}^{-1} \begin{bmatrix} E_k & 0 \\ H_k & I \end{bmatrix}$$

has n eigenvalues $\to 0$ and n that $\to \infty$. $\ker \mathcal{H}_k \approx \begin{bmatrix} I \\ -H_k \end{bmatrix}$, so $-H_k \to X$.

Dually, "$\ker \mathcal{H}_k^{-1}$" (a thing that shouldn't exist…) $\approx \begin{bmatrix} -G_k \\ I \end{bmatrix}$, so $G_k \to Y$.

Convergenza: $E_k \to 0$

$H_k \to -X$ (soluzione della CARE) stabilizzante

$G_k \to Y$

dove Y è la matrice tale che

$$\mathcal{H}\begin{bmatrix} Y \\ I \end{bmatrix} = \begin{bmatrix} Y \\ I \end{bmatrix} \hat{R}$$ è il sottosp. invariante anti-stabile (associato agli autoval. nel RHP)

La convergenza è quadratica.

(Y esiste per lo stesso motivo per cui $\begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \sim \begin{bmatrix} I \\ X \end{bmatrix}$, su $J\mathcal{H}J$).

$$\mathcal{H}\begin{bmatrix}1\\X\end{bmatrix} = \begin{bmatrix}1\\X\end{bmatrix}R \qquad \Lambda(R)\subseteq LHP$$

$$Y_0\begin{bmatrix}1\\X\end{bmatrix} = (1-\mathcal{H}_0)^{-1}(1+\mathcal{H}_0)\begin{bmatrix}1\\X\end{bmatrix} = \begin{bmatrix}1\\X\end{bmatrix}(1-R)^{-1}(1+R)$$

dove $(I-R)^{-1}(I+R) = S$ ha autoval. nel cerchio unitario, $\rho(S) < 1$

$$Y_k\begin{bmatrix}1\\X\end{bmatrix} = -(Y_0)^{2^k}\begin{bmatrix}1\\X\end{bmatrix} = \begin{bmatrix}1\\X\end{bmatrix}S^{2^k}$$

$$\begin{bmatrix}1 & G_k\\0 & F_k\end{bmatrix}^{-1}\begin{bmatrix}E_k & 0\\H_k & 1\end{bmatrix}\begin{bmatrix}1\\X\end{bmatrix} = -\begin{bmatrix}1\\X\end{bmatrix}S^{2^k} \longrightarrow$$

$$\begin{pmatrix} E_k & 0 \\ H_k & 1 \end{pmatrix}\begin{bmatrix} 1 \\ X \end{bmatrix} = \begin{bmatrix} 1 & G_k \\ 0 & f_k \end{bmatrix}\begin{bmatrix} 1 \\ X \end{bmatrix}\left(-s^{2^k}\right)$$

$$\begin{cases} E_k = (1+G_kX)\left(-s^{2^k}\right) \\ H_k + X = F_k X\left(-s^{2^k}\right) = E_k^* X\left(-s^{2^k}\right) = \end{cases}$$

$$= \left(-(s^*)^{2^k}\right)\underbrace{(1+G_kX)^{-1}X\left(-s^{2^k}\right)}_{\succcurlyeq 0}\succcurlyeq 0$$

per lo stesso motivo per cui $(1-G_kH_k)^{-1}H_k \succcurlyeq 0$

$\Rightarrow 0 \preccurlyeq -H_k \preccurlyeq X$

Se sapessi che anche G_k è limitato, $G_k \to Y$, allora posso concludere che

$$E_k = \underbrace{(1+G_k X)}_{\text{limitato}} \underbrace{(-S^{2^k})}_{\text{tende a } 0 \text{ quadraticamente}} \longrightarrow 0$$

$$H_k + X = (S^{2^k}) \cdots (-S^{2^k}) \longrightarrow 0 \text{ quadraticamente}$$

Per dimostrare che G_k limitato, si fa lo stesso ragionamento usato per dim. che H_k è limitato partendo da $\mathcal{H}\begin{bmatrix} Y \\ I \end{bmatrix} = \begin{bmatrix} Y \\ I \end{bmatrix}\hat{R}$

SDA convergence (formally)

More formally

$$\mathcal{H}_0 \begin{bmatrix} I \\ X \end{bmatrix} = (I-\mathcal{H})^{-1}(I+\mathcal{H}) \begin{bmatrix} I \\ X \end{bmatrix} = \begin{bmatrix} I \\ X \end{bmatrix} (I-\mathcal{R})^{-1}(I+\mathcal{R}).$$

where $\mathcal{S} = (I-\mathcal{R})^{-1}(I+\mathcal{R})$ has all eigenvalues in the unit circle.

$$\begin{bmatrix} I & G_k \\ 0 & E_k^* \end{bmatrix} \begin{bmatrix} I \\ X \end{bmatrix} = \begin{bmatrix} I \\ X \end{bmatrix} \begin{bmatrix} E_k & 0 \\ H_k & I \end{bmatrix} \begin{bmatrix} I \\ X \end{bmatrix} \mathcal{S}^{2^k}.$$

which implies

$$\begin{aligned} E_k &= (I + G_k X)\mathcal{S}^{2^k}, \\ H_k + X &= E_k^* X \mathcal{S}^{2^k} = (\mathcal{S}^{2^k})^*(I + XG_k)\mathcal{S}^{2^k} \succeq 0. \end{aligned}$$

The same computation on the dual equation gives $G_k \preceq Y$, so G_k is bounded and $E_k \to 0, H_k + X \to 0$ (quadratically as $\mathcal{S}^{2^k}$).