Characterization of Jordan chains

Lemma

$A(x) \in \mathbb{C}[x]^{m\times m}$ has a Jordan chain $v_0, v_1, \dots, v_{k-1}$ at $\lambda \neq \infty$ of length (exactly) k iff $(x-\lambda)^k$ divides (exactly) $A(x)v(x)$, where

$$v(x) = v_0 + v_1(x-\lambda) + v_2(x-\lambda)^2 + \cdots + v_{k-1}(x-\lambda)^{k-1} + \text{(any multiple of } (x-\lambda)^k\text{)}$$

Proof Write down $A(x)$ as a Taylor polynomial in $(x-\lambda)$:

$$A(x) = A(\lambda) + A'(\lambda)(x-\lambda) + \frac{A''(\lambda)}{2}(x-\lambda)^2 + \cdots + \frac{A^{(d)}}{d!}(x-\lambda)^d.$$

Then, the first k powers of $x-\lambda$ in the product $A(x)v(x)$ vanish $\iff$ Jordan chain conditions.

Preservation of Jordan chains

Lemma

Let $L(x), B(x) \in \mathbb{C}[x]^{m\times m}$ be matrix polynomials.

Assume that there are $E(x), F(x) \in \mathbb{C}[x]^{m\times m}$ with constant nonzero determinant such that $E(x)L(x)F(x) = B(x)$.

Then, $B(x)$ has a Jordan chain at λ of length k with associated polynomial $v(x)$ iff $L(x)$ has a Jordan chain at λ of length k with associated polynomial $F(x)v(x)$.

Proof

$$\begin{aligned}(x-\lambda)^k \mid B(x)v(x) &\implies (x-\lambda)^k \mid E(x)L(x)F(x)v(x)\\ &\implies (x-\lambda)^k \mid L(x)F(x)v(x),\end{aligned}$$

since $E(\lambda)$ is invertible. We can reverse the reasoning because an analogous polynomial relation holds $E(x)^{-1}B(x)F(x)^{-1} = L(x)$.

Linearization preserves (finite) Jordan chains

For a linearization,

$$E(x)L(x)F(x) = \operatorname{diag}(A(x), I, I, \dots, I).$$

The (finite) Jordan chains of the RHS are given by

$$\begin{bmatrix} v(x) \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

where $v(x)$ is a Jordan chain of $A(x)$ ($B(x)$ singular $\iff$ $A(x)$ singular). Hence, the Jordan chains of $L(x)$ are given by the first coefficients in the expansion in powers of $x - \lambda$ of

$$F(x) \begin{bmatrix} v(x) \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Jordan chain recovery for $C(x)$

For the Frobenius companion linearization, we can take a slight shortcut:

$$C(x)\begin{bmatrix} 1 & x & x^2 & \dots & x^{d-1} \\ & 1 & \ddots & \ddots & \vdots \\ & & \ddots & \ddots & x^2 \\ & & & 1 & x \\ & & & & 1 \end{bmatrix} = \begin{bmatrix} * & * & \cdots & * & A(x) \\ I & & & & \\ & I & & & \\ & & \ddots & & \\ & & & I & \end{bmatrix}$$

so the Jordan chains are given by expanding in powers of $x - \lambda$

$$\begin{bmatrix} 1 & x & x^2 & \dots & x^{d-1} \\ & 1 & \ddots & \ddots & \vdots \\ & & \ddots & \ddots & x^2 \\ & & & 1 & x \\ & & & & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ v_0 + v_1(x-\lambda) + \dots \end{bmatrix}.$$

(Not particularly nice to write apart from the first one.)

Remark — finiteness

- All of this is valid only for finite λ.
- For $\lambda = \infty$, there is no preservation of Jordan chains.

  Counterexample: $E(x) = I_2$, $A(x) = I_2 + 0x$, $F(x) = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}$.
- That's why we introduced strong linearizations.
- Recovery formulas for Jordan chains at ∞ follows from recovery formulas for Jordan chains at 0 for the linearization $\operatorname{rev} L(x)$ of $\operatorname{rev} A(x)$.