Example: control theory

Control theory (important subject in engineering) is the study of dynamical systems + controllers.

Example can we keep an 'inverted pendulum' in the upright position by applying a steering force?

State $x(t) = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}$, where θ is the angle formed by the pendulum (12 o' clock $\leftrightarrow$ $\theta = 0$).

Free system equations:

$$\dot{x} = \begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} x_2 \\ mg \sin x_1 \end{bmatrix} \approx \begin{bmatrix} x_2 \\ mgx_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ mg & 0 \end{bmatrix} x.$$

The system is not stable: $A = \begin{bmatrix} 0 & 1 \\ mg & 0 \end{bmatrix}$ has one positive and one negative eigenvalue.

Example: controlling an inverted pendulum

Now we apply an additional steering force u:

$$\dot{x} = Ax + Bu, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Can we choose a control $u(t)$ so that the system is stable? Yes — even better: we can choose one of the form $u(t) = Fx(t)$, $F \in \mathbb{R}^{1\times 2}$

I.e., we can literally build a contraption (engine + camera) that sets the appropriate force according to the current state only (feedback control). $u = \begin{bmatrix} f_1 & f_2 \end{bmatrix} x$ gives

$$\dot{x} = (A + BF)x = \begin{bmatrix} 0 & 1 \\ f_1 + mg & f_2 \end{bmatrix} x.$$

Choosing f_1, f_2 appropriately we can move the eigenvalues of $A + BF$ arbitrarily.

The general setup

$$\dot{x} = Ax + Bu, \quad A \in \mathbb{R}^{n\times n},\ B \in \mathbb{R}^{n\times m}.$$

Can we always stabilize a system? No — counterexample:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}.$$

No matter what we choose, we cannot change the dynamics of the second block of variables. If A_{22} has eigenvalues outside the LHP, there is nothing we can do.

Controllability / Stabilizability

This structure may be 'hidden' behind a change of basis, for instance $A \leftarrow KAK^{-1}, B \leftarrow KB$.

How do we check for it? Krylov spaces:

The pair (A, B) is called controllable if

$$\operatorname{span}(B, AB, \dots, A^k B, \dots) = \mathbb{R}^n.$$

Controllability

Definition

$(A, B) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m}$ is controllable iff $K(A, B) = \mathbb{R}^n$, where

$$K(A, B) := \operatorname{span}(B, AB, A^2B, \dots).$$

Lemma

There exists a nonsingular $M \in \mathbb{R}^{n \times n}$ such that

$$M^{-1}AM = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \quad M^{-1}B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}$$

(with $A_{11} \in \mathbb{R}^{n_1 \times n_1}$, $A_{22} \in \mathbb{R}^{n_2 \times n_2}$, $B_1 \in \mathbb{R}^{n_1 \times m}$, and $n_2 \neq 0$) if and only if (A, B) is not controllable.

Proof

$\Rightarrow$ Partition $M = \begin{bmatrix} M_1 & M_2 \end{bmatrix}$ conformably. Then,
$A^k B = M \begin{bmatrix} A_{11}^k B_1 \\ 0 \end{bmatrix} = M_1 A_{11}^k B_1$, so $K(A, B) \subseteq \operatorname{Im} M_1$.

$\Leftarrow$ Let the columns of M_1 be a basis of $K(A, B)$, and complete it to a nonsingular $M = \begin{bmatrix} M_1 & M_2 \end{bmatrix}$. Then, $M^{-1}AM$ is block triangular (because M_1 is A-invariant), and $M^{-1}B$ has zeros in the second block row (because the columns of B lie in $\operatorname{Im} M_1$).

(Linear algebra characterization: $K(A, B)$ is the smallest A-invariant subspace that contains B. It's the space Q_n that we obtain after we encounter breakdown in Arnoldi.)

Kalman decomposition

Kalman decomposition

For every matrix pair $(A, B) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m}$, there is a change of basis M such that

$$M^{-1}AM = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \quad M^{-1}B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix},$$

with (A_{11}, B_1) controllable.

Proof: as above: take M_1 such that its columns are a basis of the 'controllable space' $K(A, B)$, then complete it to a basis of $\mathbb{R}^n$.

Controllability Lyapunov equation

Let A be a stable matrix. (A, B) is controllable iff the solution of

$$AX + XA^* + BB^* = 0$$

is positive definite.

Proof $\Leftarrow$ suppose (A, B) is not controllable. Then, (up to a change of basis, Kalman decomposition)

$$\begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} X_{11} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} X_{11} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A_{11}^* & 0 \\ A_{12}^* & A_{22}^* \end{bmatrix} + \begin{bmatrix} B_1 B_1^* & 0 \\ 0 & 0 \end{bmatrix} = 0.$$

where X_{11} solves $A_{11}X_{11} + XA_{11}^* + B_1B_1^* = 0$.

$\Rightarrow$ Suppose (A, B) is controllable. Then, for each $v \neq 0$, $v^* A^k B$ is not zero for all $k \implies v^* e^{At} B$ is not zero for all $t \implies v^* X v = \int v^* e^{At} B B^* e^{A^* t} v \mathrm{d}t \neq 0$.

Indeed, if $v^* \exp(tA) B = 0$ for all t, then $v^* B = 0$ (taking $t = 0$),

$$\lim_{t \to 0} \frac{1}{t} v^* (\exp(tA) - I) B = v^* A b = 0,$$

$$\lim_{t \to 0} \frac{1}{t^2} v^* (\exp(tA) - I - tA) B = v^* \frac{1}{2} A^2 b = 0,$$

$$\vdots$$

Bass algorithm

Let $\alpha > \rho(A)$; then $A + \alpha I$ has eigenvalues in the RHP, and the Lyapunov equation

$$(A + \alpha I)X + X(A + \alpha I)^* = 2BB^*$$

has a solution $X \succeq 0$.

By the previous lemma, $X \succ 0$ (indeed, $K(A + \alpha I, B) = K(A, B)$).

Then,

$$(A - BB^*X^{-1})X + X(A - BB^*X^{-1})^* = -2\alpha X,$$

which proves that $A - B(B^*X^{-1})$ has eigenvalues in the LHP.

(Actually, if (A, B) is controllable, we can find F such that $A + BF$ has any chosen spectrum.)

Stabilizability

Weaker condition: sometimes even if a system is not controllable we can still ensure it's stable via a feedback control.

Definition

(A, B) is stabilizable if in its Kalman decomposition A_{22} is stable (i.e., $\Lambda(A_{22}) \subseteq LHP$).

Note that this definition is well-posed even if M is non-unique: the eigenvalues of A_{11} are the eigenvalues of $A|_{K(A,B)}$, and those of A_{22} are the remaining eigenvalues of A (counting with their algebraic multiplicity).