Recap: control theory

Study of linear systems of ODEs with control functions $u(t)$

$$\dot{x}(t) = Ax(t) + Bu(t),$$

$A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$.

Goal: choose a feedback control $u(t) = Fx(t)$ such that the closed-loop system $\dot{x}(t) = (A + BF)x(t)$ is stable, i.e.,
$\operatorname{eig}(A + BF) \subseteq LHP$.
Always solvable? No: counterexample:

$$\hat{A} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \quad \hat{B} = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}. \quad \dot{x}_2 = A_{22}\dot{x} + 0 \cdot u$$

May be hidden behind a change of basis for the state variable,
$My = x$,

$$A = M\hat{A}M^{-1}, \quad B = M\hat{B}.$$

$\dot{x} = Ax + Bu$

$\dot{y} = \hat{A}y + \hat{B}u$

Controllability

The pair (A, B) is called controllable if one of the following (equivalent) conditions holds:

- No invertible M gives $M^{-1}AM = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$, $M^{-1}B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}$.
- $K(A, B) = \text{span}(B, AB, \ldots, A^{n-1}B) = \mathbb{R}^n$
- The Lyapunov equation $AX + XA^* + BB^* = 0$ has solution $X \succ 0$. (se eig(A) ⊆ LHS)

$$X = \int_0^\infty \exp(At) BB^* \exp(A^* t)\, dt$$

$X \succcurlyeq 0$ $\quad X \succ 0$

- $\text{rank}\,[sI - A, B] = n$ per ogni $s \in \mathbb{C}$

Controllability: further conditions

Further equivalent conditions:

- $\begin{bmatrix} A - sI & B \end{bmatrix} \in \mathbb{R}^{n \times (n+1)}$ has full rank n for all $s \in \mathbb{C}$.

Proof: only nontrivial case when s is an eigenvalue of A.
Controllable iff $u^* B \neq 0$ for each left eigenvector u of A.

And the following ones which we don't prove here.

- For any $t > 0$ and $\hat{x} \in \mathbb{R}^n$, one can choose the control function u so that $x(t) = \hat{x}$.
- One can choose F so that $\mathrm{eig}(A + BF)$ is any prescribed set of n eigenvalues.

$$M^{-1}AM = \begin{bmatrix} & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \\ \times & \times & \times & \times & \times & \times \end{bmatrix}, \quad M^{-1}B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

Bass algorithm

We just show an easier trick: if (A, B) is controllable, compute a stabilizing control.

Let $\alpha > \rho(A)$; then $A + \alpha I$ has eigenvalues in the RHP, and the Lyapunov equation

$$(A + \alpha I)X + X(A + \alpha I)^* = 2BB^*$$

has a solution $X \succeq 0$.

By the previous lemma, $X \succ 0$ (note $K(A + \alpha I, B) = K(A, B)$).

Then,

$$(A - BB^* X^{-1})X + X(A - BB^* X^{-1})^* = -2\alpha X,$$

which proves that $A - B(B^* X^{-1})$ has eigenvalues in the LHP.

Kalman decomposition

Kalman decomposition

For every matrix pair $(A, B) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m}$, there is a change of basis M such that

$$M^{-1}AM = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \quad M^{-1}B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix},$$

with (A_{11}, B_1) controllable.

Proof: as above: take M_1 such that its columns are a basis of the 'controllable space' $K(A, B)$, then complete it to a basis of $\mathbb{R}^n$.

Stabilizability

Weaker condition: sometimes even if a system is not controllable we can still ensure it's stable via a feedback control.

Definition

(A, B) is stabilizable if in its Kalman decomposition A_{22} is stable (i.e., $\Lambda(A_{22}) \subseteq LHP$).

Note that this definition is well-posed even if M is non-unique: the eigenvalues of A_{11} are the eigenvalues of $A|_{K(A,B)}$, and those of A_{22} are the remaining eigenvalues of A (counting with their algebraic multiplicity).

(A, B) stabilizable iff rank $\begin{bmatrix} A - sI & B \end{bmatrix} = n$ for all $s \notin LHP$.