

Invariant subspace methods for CAREs

X solves CARE $A^*X + XA + Q = XGX$ iff

$$\begin{bmatrix} A & -G \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} I \\ X \end{bmatrix} = \begin{bmatrix} I \\ X \end{bmatrix} \mathcal{R}, \quad \mathcal{R} = A - GX.$$

One can find X through an invariant subspace of the Hamiltonian.

```
>> [A,G,Q] = carex(4) %if test suite is installed
>> n = length(A);
>> H = [A -G; -Q -A'];
>> [U, T] = schur(H);
>> [U, T] =ordschur(U, T, 'lhp');
>> X = U(n+1:2*n, 1:n) / U(1:n, 1:n);
```

People are not satisfied with this method though — it is not **structured** backward stable.

Eigenvalues close to the imaginary axis can be 'mixed up' — try `carex(14)` for instance.

Symplectic transformations

Ideal setting: make transformations at each step that are orthogonal **and** symplectic, i.e., orthogonal w.r.t the scalar product

$$J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}: \text{ they satisfy } S^T J S = J.$$

For instance:

- ▶ If $Q \in \mathbb{R}^{n \times n}$ is any orthogonal matrix, then $\text{blkdiag}(Q, Q)$ is orthogonal and symplectic.
- ▶ A Givens matrix that acts on entries k and $n+k$ (i.e., $G = \text{eye}(2*n)$; $G([k, n+k], [k, n+k]) = [c \ s; -s \ c]$;) is orthogonal and symplectic.

Laub trick: let $U = \begin{bmatrix} U_1 & U_3 \\ U_2 & U_4 \end{bmatrix}$ the unitary matrix produced by $\text{schur}(H)$. Then, $\begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$ is an orthogonal matrix that spans the stable subspace. We know that $-J \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = \begin{bmatrix} U_2 \\ -U_1 \end{bmatrix}$ is orthogonal to it (and spans the left unstable invariant subspace).

It turns out that $V = \begin{bmatrix} U_1 & U_2 \\ U_2 & -U_1 \end{bmatrix}$ is orthogonal **and** symplectic, and $V^T \mathcal{H} V = \begin{bmatrix} R & S \\ 0 & -R^* \end{bmatrix}$, with R upper triangular and S symmetric (**Hamiltonian Schur form**).

An orthogonal symplectic algorithm

This produces the same subspace as the previous method, so it is not really a 'structured' method. Can one do a 'symplectic QR' and compute the Hamiltonian Schur form using a sequence of orthosymplectic transformations?

Open problem for a while; it turns out that the Hamiltonian Schur form does not exist for all Hamiltonian matrices (there are counterexamples with eigenvalues on the unit circle). \implies algorithms must be unstable for nearby matrices.

(This problem was known as **Van Loan's curse**.)

Chu–Liu–Mehrmann algorithm

Closest thing to a solution: Chu–Liu–Mehrmann algorithm. Based on a different decomposition: $\mathcal{H} = URV^T$, with U, V orthosymplectic and

$$R = \begin{bmatrix} R_{11} & R_{12} \\ 0 & R_{22} \end{bmatrix}$$

with R_{11}, R_{22}^* upper triangular.

Can be computed 'almost' directly in $O(n^3)$ (it's an LU-like decomposition).

URV — simpler version (produces Hessenberg R_{22})

- ▶ Left-multiply by $\text{blkdiag}(Q, Q)$ to get

```

* * * * *
* * * * *
* * * * *
* * * * *
0 * * * *
0 * * * *
    
```

- ▶ Left-multiply by a Givens on $(1, n + 1)$ to get

```

* * * * *
* * * * *
* * * * *
0 * * * *
0 * * * *
0 * * * *
    
```

- ▶ Left-multiply by $\text{blkdiag}(Q, Q)$ to get

```

* * * * *
0 * * * *
0 * * * *
0 * * * *
0 * * * *
0 * * * *
    
```

- ▶ Right-multiply by $\text{blkdiag}(Q, Q)$ to get

```

* * * * *
0 * * * *
0 * * * *
0 * * * *
0 * * * *
0 * * * *
    
```

- ▶ Right-multiply by a Givens on $(2, n + 2)$ to get

```

* * * * *
0 * * * *
0 * * * *
0 * * * *
0 * * * *
0 * * * *
    
```

- ▶ Right-multiply by $\text{blkdiag}(Q, Q)$ to get

```

* * * * *
0 * * * *
0 0 0 * * 0
0 * * * *
0 * * * *
    
```

Using URV

Note that $\mathcal{H} = URV$ + symplecticity implies

$$\mathcal{H} = V \begin{bmatrix} -R_{22}^T & R_{12}^T \\ 0 & -R_{11}^T \end{bmatrix} U^T.$$

Hence

$$\mathcal{H}^2 = V \begin{bmatrix} -R_{11}R_{22}^T & * \\ 0 & -R_{22}R_{11}^T \end{bmatrix} V^T$$

can be used to compute eigenvalues (easily) and eigenvectors of \mathcal{H} (for instance: the columns of V cause breakdown at step 2 in Arnoldi).