Note Title 2020-04-02

Teo: la derivata di Fréchet di $f(A)$ $L_{f,A}$

ha autovalori

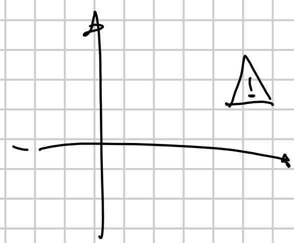
$$i,j=1,\dots,n \qquad f[\lambda_i,\lambda_j]=\begin{cases}\dfrac{f(\lambda_i)-f(\lambda_j)}{\lambda_i-\lambda_j} & \text{se } i\neq j\\ f'(\lambda_j) & \text{se } i=j\end{cases}$$

($\lambda_1,\lambda_2,\dots\lambda_n$ autovalori di A)

ES: Quanto $K_{abs,\exp,A}$ è grande?

per $f=\exp$, $f'(x)=\exp(x)$ $\quad f'(\lambda_j)$ grande quando $\exp(\lambda_j)$ è grande

$$\frac{f'(\lambda_i)-f'(\lambda_j)}{\lambda_i-\lambda_j}=f'(\xi) \text{ per uno } \xi$$
sul segmento $[\lambda_i,\lambda_j]$

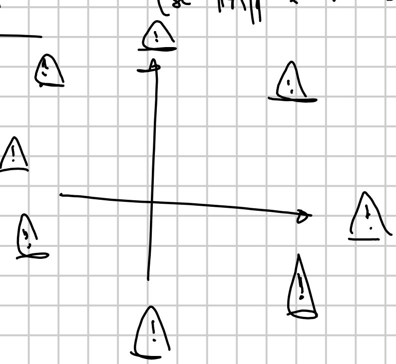


Autovalori a dx = K_{abs} grande

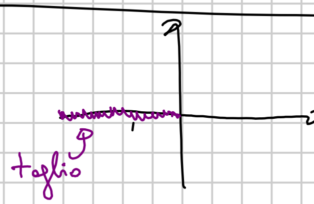
$$K_{rel,\exp,A}=K_{abs,\exp,A}\cdot\frac{\|A\|_F}{\|f(A)\|_F} \qquad \|A\|_F \approx \sqrt{\sum|\lambda_i|^2} \qquad \|f(A)\|_F \approx \sqrt{\sum|\exp(\lambda_i)|^2}$$

$$\approx \frac{\max_i|\exp(\lambda_i)|\sqrt{\sum|\lambda_i|^2}}{\sqrt{\sum|\exp(\lambda_i)|^2}} \geq 1$$

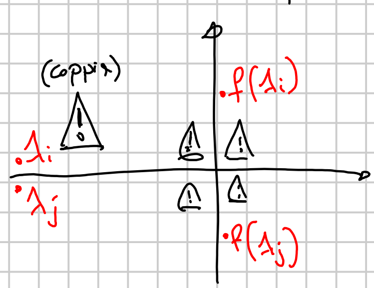
(se $\|A\|$ è normale o quasi)



Stessa cosa per $f(x)=x^{\frac{1}{2}}$ (ramo principale)

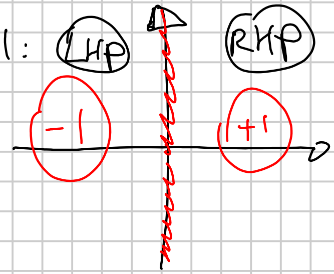


$$f'(x)=\frac{1}{2}x^{-\frac{1}{2}}$$



In più, $\dfrac{f(\lambda_i)-f(\lambda_j)}{\lambda_i-\lambda_j}$ è grande se lo due autovalori vicini da parti opposte del taglio

---

We study another function in detail: 

left / right half-plane

$$\operatorname{sign}(x)=\begin{cases}1 & \operatorname{Re} x>0\\ -1 & \operatorname{Re} x<0\\ \text{undefined} & \operatorname{Re} x=0\end{cases}$$

$$A=\left[\begin{array}{c|c} V_1 & V_2\end{array}\right]\left[\begin{array}{c|c} J_1 & 0\\ \hline 0 & J_2\end{array}\right]\left[\begin{array}{c|c} V_1 & V_2\end{array}\right]^{-1}$$

Jordan form,
J_1 contains eigenvalues in LHP
J_2 " " " RHP

$$f(A)=V f(J) V^{-1}=V\left[\begin{array}{c|c} f(J_1) & 0\\ \hline 0 & f(J_2)\end{array}\right]V^{-1}=V\left[\begin{array}{c|c} -I & 0\\ \hline 0 & I\end{array}\right]V^{-1}$$

V_1 = stable invariant subspace = span(Jordan chains with eigvl. in LHP)

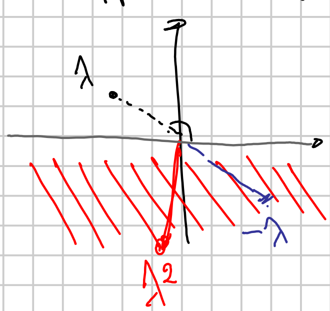
V_2 = antistable inv. subspace = span(Jordan chains in RHP)

$$V_1=\ker\left(\operatorname{sign}(A)+I\right)\qquad V_2=\ker\left(\operatorname{sign}(A)-I\right)$$

Formula: $\boxed{\operatorname{sign}(A)=A\left(A^2\right)^{-1/2}}$ $\left(M^{1/2}=\text{principal sqrt of } M\right)$

Let us see what happens to eigenvalues:

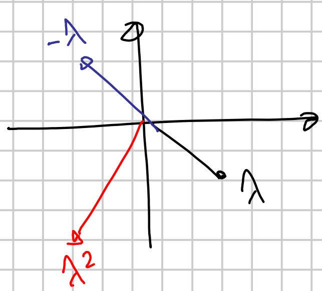
$\lambda\in LHP$



$\lambda\in LHP \Rightarrow \left(\lambda^2\right)^{1/2}=-\lambda$

$\left(\lambda^2\right)^{1/2}=-\lambda$

$\operatorname{sign}(\lambda)=\lambda\cdot\dfrac{1}{-\lambda}=-1$ ✓



$\left(\lambda^2\right)^{1/2}=\lambda$ $1=\operatorname{sign}(\lambda)=\lambda\cdot\dfrac{1}{\lambda}$ ✓

$\Rightarrow \operatorname{sign}(x)=x\left(x^2\right)^{-1/2}$ for scalars

$\Rightarrow \operatorname{sign}(A)=A\left(A^2\right)^{-1/2}$ for A with distinct eigenvalues

$\Rightarrow \operatorname{sign}(A) = A\left(A^2\right)^{-1/2}$ for all A with eigenvalues outside $i\mathbb{R}$, by continuity

---

<u>Theorem</u>: if AB has no eigenvalues in $\mathbb{R}_{\leq 0}$, then

$$(*) \quad \operatorname{sign}\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} = \begin{bmatrix} 0 & C \\ C^{-1} & 0 \end{bmatrix} \qquad C = A(BA)^{-1/2}$$

In particular, when $B = I$

$$\operatorname{sign}\begin{bmatrix} 0 & A \\ I & 0 \end{bmatrix} = \begin{bmatrix} 0 & A^{1/2} \\ A^{-1/2} & 0 \end{bmatrix}$$

<u>Remark</u>: $\operatorname{eig}(AB) = \operatorname{eig}(BA)$, so $(BA)^{-1/2}$ exists

$$(*) \quad \operatorname{sign}\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} = \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} \cdot \left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}^2\right)^{-1/2}$$

$$= \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} \cdot \left(\begin{bmatrix} AB & 0 \\ 0 & BA \end{bmatrix}\right)^{-1/2} = \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\begin{bmatrix} (AB)^{-1/2} & 0 \\ 0 & (BA)^{-1/2} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \boxed{A(BA)^{-1/2}} \\ B(AB)^{-1/2} & 0 \end{bmatrix} \quad C \ \checkmark$$

Note that $(\operatorname{sign}(M))^2 = I$ for any M

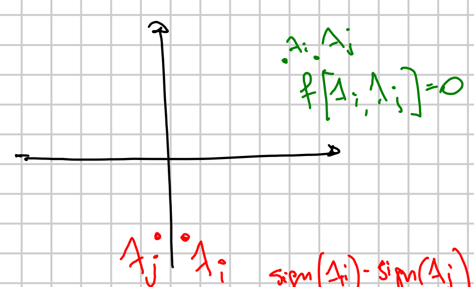
$$\overset{?}{=}$$

$$C^{-1} \overset{?}{=} D = \left(A(BA)^{-1/2}\right)^{-1}$$

$$\text{so } \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & C \\ D & 0 \end{bmatrix}^2 = \begin{bmatrix} CD & 0 \\ 0 & DC \end{bmatrix} \Rightarrow \quad CD = DC = I$$

Conditioning of sign:

$$\operatorname{eig}(L_{\operatorname{sign},X}) = \begin{cases} \dfrac{\operatorname{sign}(\lambda_i) - \operatorname{sign}(\lambda_j)}{\lambda_i - \lambda_j} & i \neq j \\ 0 & i = j \end{cases}$$



$$\frac{\operatorname{sign}(\lambda_i) - \operatorname{sign}(\lambda_j)}{\lambda_i - \lambda_j} = \frac{2}{\lambda_i - \lambda_j}$$

<u>Theorem</u>: $K_{\text{abs},\operatorname{sign},A} = \left\| \left(I \otimes N + N^T \otimes I\right)^{-1}\left(I_{n^2} - S^T \otimes S\right) \right\|_2$

$S = \operatorname{sign}(A) \qquad N = (A^2)^{1/2} \quad (\text{so } S = AN^{-1})$

(recall that $\|(I \otimes N + N^T \otimes I)^{-1}\| = \frac{1}{\sigma_{\min}(I \otimes N + N^T \otimes I)} = \frac{1}{\operatorname{sep}(N, -N)}$)

$$\begin{bmatrix} N & * \\ 0 & -N \end{bmatrix} \longrightarrow \begin{bmatrix} N & 0 \\ 0 & -N \end{bmatrix}$$

If all eigenvalues of A are on the same side of the imag. axis (for instance $\Lambda(A) \subseteq LHP$) then $S = \operatorname{sign}(A) = \begin{bmatrix} V_1 & \vdots \end{bmatrix} \begin{bmatrix} -I & \vdots \\ \cdots & \vdots \end{bmatrix} \begin{bmatrix} V_1 & \vdots \end{bmatrix}^{-1} = -I$

and $\kappa_{abs, \operatorname{sign}, A} = 0$ from the above formula. Not surprising because $\operatorname{sign}(A) = -I$ for all matrices with eigvls in LHP, so also for small perturbations.

Numerical methods for sign

One idea: Schur-Parlett: 1) $A = QTQ^*$ (Schur decomp.)

2) $f(A) = Q f(T) Q^*$, $f(T)$ computed using

$$f(T) = \begin{bmatrix} f(t_{11}) & & * \\ & f(t_{22}) & s_{ij} \\ 0 & & \ddots \\ & & & f(t_{nn}) \end{bmatrix}$$

using recurrence formulas like

$$s_{i,i+1} = t_{i,i+1} \frac{f(t_{i+1,i+1}) - f(t_{ii})}{t_{i+1,i+1} - t_{ii}}$$

For the sign: you can reorder the schur decomp. such that

$$Q^* A Q = \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix} \qquad \begin{array}{l} \Lambda(T_{11}) \subseteq LHP \\ \Lambda(T_{22}) \subseteq RHP \end{array}$$

$$f(T) = \begin{bmatrix} \overset{-I}{\overset{\shortparallel}{f(T_{11})}} & X \\ 0 & \underset{I}{\underset{\shortparallel}{f(T_{22})}} \end{bmatrix}$$

X solves (obtained from $f(T)T = Tf(T)$)

$$\boxed{T_{11}X - XT_{22} = -T_{12} f(T_{22}) + f(T_{11}) T_{12} = -2T_{12}}$$

1) Compute $A = QTQ^*$

2) Reorder to obtain $\hat{Q}\hat{T}\hat{Q}^* = A$, with $\hat{T} = \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix}$ $\begin{array}{l} \Lambda(T_{11}) \subseteq LHP \\ \Lambda(T_{22}) \subseteq RHP \end{array}$

3) Solve $T_{11}X - XT_{22} = 2T_{12}$ (condition number depends on $\operatorname{sep}(T_{11}, T_{22})$)

4) $f(A) = \hat{Q} \begin{bmatrix} -I & X \\ 0 & I \end{bmatrix} \hat{Q}^*$

Check for the Sylv. equation:

$$\begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix}\begin{bmatrix} -1 & X \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & X \\ 0 & 1 \end{bmatrix}\begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix}$$

$$T_{11}X + T_{12} = -T_{12} + XT_{22}$$

$$T_{11}X - XT_{22} + 2T_{12} = 0$$

More popular algorithm: Newton iteration (for the matrix sign)

$$X_0 = A \qquad X_{k+1} = \frac{1}{2}\left(X_k + X_k^{-1}\right) \qquad k = 0, 1, 2, \dots$$

Then, $\lim_{k \to \infty} X_k = S$ (if A diagonalizable)

We can reduce it to a scalar iteration: $A = V \Lambda V^{-1} = X_0$

$$X_1 = \frac{1}{2}\left(V\Lambda V^{-1} + V\Lambda^{-1}V^{-1}\right) = V \underbrace{\begin{bmatrix} \frac{1}{2}(\lambda_1 + \lambda_1^{-1}) & & & \\ & \frac{1}{2}(\lambda_2 + \lambda_2^{-1}) & & \\ & & \ddots & \\ & & & \frac{1}{2}(\lambda_n + \lambda_n^{-1}) \end{bmatrix}}_{\begin{bmatrix} \lambda_1^{(1)} & & \\ & \ddots & \\ & & \lambda_n^{(1)} \end{bmatrix}} V^{-1}$$

$$X_2 = \frac{1}{2}\left(V\Lambda^{(1)}V^{-1} + V\Lambda^{(1)-1}V^{-1}\right) = V \begin{bmatrix} \frac{1}{2}(\lambda_1^{(1)} + \lambda_1^{(1)-1}) & & & \\ & \frac{1}{2}(\lambda_2^{(1)} + \lambda_2^{(1)-1}) & & \\ & & \ddots & \\ & & & \frac{1}{2}(\lambda_n^{(1)} + \lambda_n^{(1)-1}) \end{bmatrix} V^{-1}$$

$\Leftrightarrow$ running on each eigenvalue the scalar iteration

$(*)$ $\quad x_0 = \lambda \qquad x_{k+1} = \frac{1}{2}\left(x_k + x_k^{-1}\right) = \dfrac{x_k^2 + 1}{2x_k}$

Fixed points: y s.t.

$$y = \frac{y^2 + 1}{2y} \qquad 2y^2 = y^2 + 1 \quad y^2 = 1 \Leftrightarrow y = \pm 1$$

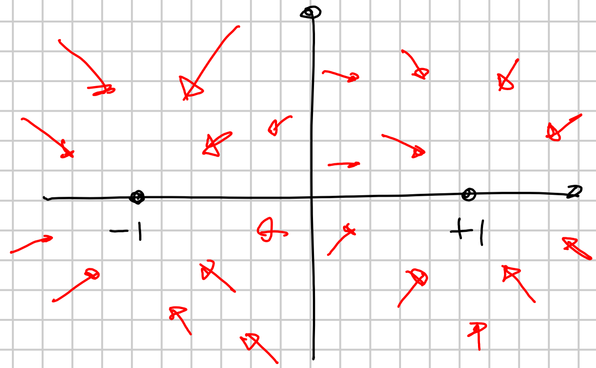
Two more properties: (1) quadratically convergent

(2) if $x_k \in LHP \Rightarrow x_{k+1} \in LHP$

$x_k \in RHP \Rightarrow x_{k+1} \in RHP$

$(*)$ is Newton's method for $f(x) = x^2 - 1$:

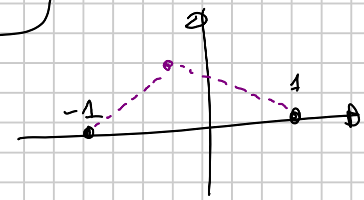
$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} = x_k - \frac{x_k^2 - 1}{2x_k} = \frac{2x_k^2 - x_k^2 + 1}{2x_k} = \frac{x_k^2 + 1}{2x_k}$$



Proof (behaviour of scalar iteration): (Cayley transform)

$$x \qquad y = \frac{1+x}{1-x}$$

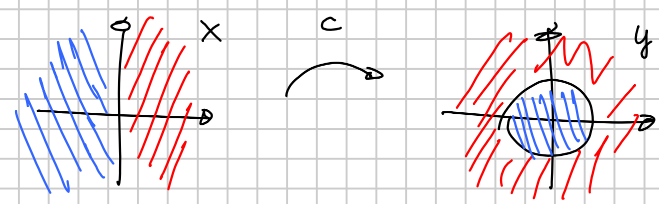
$$x = \frac{y-1}{y+1}$$



If $x \in LHP \Rightarrow x$ closer to -1 than to $1 \Rightarrow |x+1| < |x-1|$

$$\Rightarrow |y| = \left|\frac{1+x}{1-x}\right| < 1$$

Symmetrically, $x \in RHP \rightarrow \qquad |y| > 1$



(exactly like the exponential)

$$x_{k+1} = \frac{1}{2}\left(x_k + \frac{1}{x_k}\right) \qquad y_k = \frac{1+x_k}{1-x_k}, \quad y_{k+1} = \frac{1+x_{k+1}}{1-x_{k+1}} \qquad \Longleftrightarrow \qquad y_{k+1} = -y_k^2$$

$$y_{k+1} = \frac{1+x_{k+1}}{1-x_{k+1}} = \frac{\left(1+\frac{1}{2}\left(x_k + \frac{1}{x_k}\right)\right)\cdot 2x_k}{\left(1-\frac{1}{2}\left(x_k + \frac{1}{x_k}\right)\right)\cdot 2x_k} = \frac{2x_k + x_k^2 + 1}{2x_k - x_k^2 - 1} = -\left(\frac{1+x_k}{1-x_k}\right)^2$$

$$= -y_k^2$$

So, if $x_0 \in LHP$, $y_0 \in$ unit disk $\Rightarrow -y_0^2, -y_0^4, -y_0^8, \ldots$ are inside the unit circle and $\rightarrow 0$

and, symmetrically, $x_0 \in RHP$, $|y_0| > 1 \Rightarrow -y_0^2, -y_0^4, -y_0^8 \ldots$ diverge to ∞.