Lyapunov equations

Lyapunov equation

$$A^*X + XA + Q = 0, \quad Q = Q^* \succeq 0. \tag{L}$$

Special case of the Sylvester equation.
It has unique solutions whenever $\Lambda(A) \cup \Lambda(-A^*) = \emptyset$.
Important case: when $\Lambda(A) \subseteq LHP$ (open).

Lemma

Suppose (L) has a unique solution X; then X is Hermitian.

Proof: transpose everything; X^* is another solution.

Lyapunov equation: positivity

Lemma

Suppose A has eigenvalues in the (open) LHP. Then, $Q \succeq 0$ implies $X \succeq 0$, and $Q \succ 0$ implies $X \succ 0$.

Proof Check that

$$X = \int_0^\infty e^{A^* t} Q e^{At} \, \mathrm{d}t$$

($\frac{\mathrm{d}}{\mathrm{d}t} e^{A^* t} Q e^{At} = A^* e^{A^* t} Q e^{At} + e^{A^* t} Q e^{At} A$, then integrate both sides).

Lemma

Suppose $Q \succ 0$ and $X \succ 0$. Then, A has eigenvalues in the (open) LHP.

Proof Let $Av = \lambda v$; then $0 < v^* Q v = \dots$

Alternative statement: A has all its eigenvalues in the LHP if there exists $X \succ 0$ such that $A^* X + XA \prec 0$.

Lyapunov's version

Alternative way to see it: the dynamical system $\dot{x}(t) = Ax(t)$ is stable (i.e., $\lim_{t\to\infty} x(t) = 0$) for all initial values x_0 if and only if there exists $X \succ 0$ such that $A^*X + XA \prec 0$.

Lyapunov's proof: $V(t) = x^*Xx$ ('energy function') is such that $\frac{d}{dt}V(x(t)) \leq 0$, so $x(t)$ cannot escape the (finite) region $\{x \colon V(x) \leq V(x_0)\}$.

Lyapunov (1857–1918) did not have computers, so at that time exhibiting a solution to that equation was typically easier than computing all the eigenvalues of a nonsymmetric matrix A.

Discrete-time version

Many of these results also come in a 'discrete-time' variant; in this case:

Stein's equation

A has all its eigenvalues in the (open) unit disc iff

$$X - A^* X A = Q, \quad Q \succ 0$$

has a solution $X \succ 0$.

Deals with stability of the discrete-time system $x_{t+1} = A x_t$.

Can be solved with a Bartels-Stewart-like method.
(Skipping some details: a certain quantity has to be cached during the computation so that it is feasible in $O(n^3)$; you can think about it yourself!)
(Actually B-S works for all equations of the kind $AXB + CXD = E$, using QZ factorizations of (A, C) and (D^T, B^T).)

Discrete-time version

Discrete-time version of the integral formula:

If A has all its eigenvalues inside the (open) unit disc then

$$X = \sum_{k=0}^{\infty} (A^*)^k Q A^k.$$

Proof $(I - A^T \otimes A^*)\,\mathrm{vec}(X) = \mathrm{vec}(Q)$, series for the inverse.