Optimal control

Several choices available for stabilizing feedback F: for instance, you can choose different α's in Bass algorithm.

Is there an 'optimal' one?

Linear-quadratic optimal control

Find $u : [0, \infty] \to \mathbb{R}$ (piecewise C^0, let's say) that minimizes

$$E = \int_0^\infty x^* Q x + u^* R u \, \mathrm{d}t$$

$$\text{s.t. } \dot{x} = Ax + Bu,\ x(0) = x_0.$$

Minimum 'energy' defined by a quadratic form ($R \succeq 0$, $Q \succeq 0$).

We assume $R \succ 0$: control is never free. Trickier problem otherwise.

Optimal control — solution

Using calculus of variations tools, one can prove this result.

Pontryagin's maximum principle

A pair of functions u, x solves the optimal control problem iff there exists a function $\mu(t)$ ('Lagrange multiplier') such that

$$\begin{bmatrix} 0 & I & 0 \\ -I & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\mu} \\ \dot{x} \\ \dot{u} \end{bmatrix} = \begin{bmatrix} 0 & A & B \\ A^T & Q & 0 \\ B^T & 0 & R \end{bmatrix} \begin{bmatrix} \mu \\ x \\ u \end{bmatrix},$$

$x(0) = x_0$, $\lim_{t \to \infty} \left[\begin{smallmatrix} \mu \\ x \\ u \end{smallmatrix}\right] = 0$.

We would know how to solve this if the matrix in the LHS were invertible. Unfortunately it is not!

There are techniques to solve this problem directly working on the two matrices, but they involve some matrix pencil theory that we did not treat in this course.

Change of variables

How to get rid of the last equation? Back-substitution. μ, x, u solve

$$\begin{bmatrix} 0 & I & 0 \\ -I & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\mu} \\ \dot{x} \\ \dot{u} \end{bmatrix} = \begin{bmatrix} 0 & A & B \\ A^T & Q & 0 \\ B^T & 0 & R \end{bmatrix} \begin{bmatrix} \mu \\ x \\ u \end{bmatrix}$$

iff $u = -R^{-1}B^T\mu$ and μ, x solve

$$\begin{bmatrix} & I \\ -I & \end{bmatrix} \begin{bmatrix} \dot{\mu} \\ \dot{x} \end{bmatrix} = \begin{bmatrix} -BR^{-1}B^T & A \\ A^T & Q \end{bmatrix} \begin{bmatrix} \mu \\ x \end{bmatrix}$$

The problem

$$\begin{bmatrix} \dot{x} \\ \dot{\mu} \end{bmatrix} = \mathcal{H} \begin{bmatrix} x \\ \mu \end{bmatrix}, \quad \mathcal{H} = \begin{bmatrix} A & -G \\ -Q & -A^T \end{bmatrix}, \quad G = BR^{-1}B^T.$$

with

$$x(0) = x_0, \quad \lim_{t\to\infty} \begin{bmatrix} x(t) \\ \mu(t) \end{bmatrix} = 0.$$

Solving linear differential equations

The general solution of $\dot{v} = \mathcal{H}v$ is

$$v(t) = \exp(t\mathcal{H})v_0.$$

Plugging in a Jordan decomposition of $\mathcal{H}$,

$$v(t) = V \begin{bmatrix} \exp(tJ_1) & & \\ & \ddots & \\ & & \exp(tJ_s) \end{bmatrix} (V^{-1}v_0).$$

Which of these are stable ($\lim_{t\to\infty} v(t) = 0$)? Entries of $V^{-1}v_0$ that match Jordan blocks with $\lambda \notin LHP$ must be 0.

I.e., we must have $v_0 \in \mathcal{U}$, where $\mathcal{U}$ is the stable invariant subspace of $\mathcal{H}$

$$\mathcal{U} = \operatorname{span}\{v_i \colon v_i \text{ belongs to a Jordan chain of } \mathcal{H} \text{ with } \lambda \in LHP\}.$$

Solving the reduced problem

Suppose that:

- $\mathcal{H}$ has n eigenvalues in the LHP and n in the RHP.
- we find X such that $\begin{bmatrix} I \\ X \end{bmatrix}$ spans the stable invariant subspace of $\mathcal{H}$, i.e., $\mathcal{H} \begin{bmatrix} I \\ X \end{bmatrix} = \begin{bmatrix} I \\ X \end{bmatrix} \mathcal{R}$, $\Lambda(R) \subset LHP$.

Then, the stable solutions of

$$\begin{bmatrix} \dot{x} \\ \dot{\mu} \end{bmatrix} = \mathcal{H} \begin{bmatrix} x \\ \mu \end{bmatrix}$$

are given by

$$\begin{bmatrix} x(t) \\ \mu(t) \end{bmatrix} = \begin{bmatrix} I \\ X \end{bmatrix} \exp(\mathcal{R}t)v.$$

The initial condition $x(0) = x_0$ gives $v = x_0$. Moreover, $\mu(t) = Xx(t)$, hence $u(t) = -R^{-1}B^T Xx(t)$.

Algebraic Riccati equations

We have reduced the problem to $\mathcal{H}\begin{bmatrix} I \\ X \end{bmatrix} = \begin{bmatrix} I \\ X \end{bmatrix}\mathcal{R}$, or

$$\begin{bmatrix} A & -G \\ -Q & -A^T \end{bmatrix}\begin{bmatrix} I \\ X \end{bmatrix} = \begin{bmatrix} I \\ X \end{bmatrix}\mathcal{R}$$

$$\mathcal{R} = A - GX, \quad -Q - A^T X = XA - XGX.$$

$$A^T X + XA + Q - XGX = 0, \quad Q \succeq 0, \; G \succeq 0$$

is called algebraic Riccati equation. We look for a stabilizing solution, i.e., $\Lambda(\mathcal{R}) \subseteq LHP$.

This formulation as a matrix equation is alternative to the invariant subspace one. It was preferred (and still is, to some extent) by the engineers.

Linear-quadratic regulator theorem [Datta, Thm 10.5.1]

A direct proof of the optimality result using the Riccati formulation.

Theorem

Let $Q \succeq 0$, $R \succ 0$, $G = BR^{-1}B^T \succeq 0$. Suppose that there exists X with

- $A^T X + XA + Q - XGX = 0$, (X solves ARE),
- $A - GX \succeq 0$,
- $X \succeq 0$.

Then, the solution of the minimum problem

$$\min \int_0^\infty x(t)^T Q x(t) + u(t)^T R u(t) \, \mathrm{d}t,$$
$$\text{s.t. } \dot{x}(t) = Ax(t) + Bu(t), \quad \lim x(t) \to 0$$

is $x_0^T X x_0$, attained when $u(t) = -R^{-1}B^T X x(t)$ for all t.

Proof

Proof

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} x^T X x &= \dot{x}^T X x + x^T X \dot{x} \\
&= (Ax + Bu)^T X x + x^T X (Ax + Bu) \\
&= x^T (A^T X + XA) x + u^T B^T X x + x^T X B u \\
&= x^T (X B R^{-1} B^T X - Q) x + u^T B^T X x + x^T X B u \\
&= \underbrace{(u + R^{-1} B^T X x)^T R (u + R^{-1} B^T X x)}_{\succeq 0} - x^T Q x - u^T R u.
\end{aligned}$$

Integrating from 0 to ∞,

$$\int_0^\infty x^T Q x + u^T R u \, \mathrm{d}t \geq x_0^T X x_0 - \underbrace{x(\infty)^T X x(\infty)}_{=0},$$

with equality if $u + R^{-1} B^T X x \equiv 0$.

Solvability conditions

Solutions of (ARE) $\iff$ n-dimensional invariant subspaces of $\mathcal{H}$ with invertible top block.

If $\mathcal{H}$ has distinct eigenvalues, there are at most $\binom{2n}{n}$ solutions (choose n eigenvalues out of the $2n\dots$)

Solvability conditions

Does the ARE have a (unique) stabilizing solution? For this, $\mathcal{H}$ Must have (exactly) n eigenvalues in the LHP, and the associated invariant subspace must be expressible as $\operatorname{Im}\begin{bmatrix} I \\ X \end{bmatrix}$.

Next goal: show that these assumptions hold.

Hamiltonian matrices

$$\mathcal{H} = \begin{bmatrix} A & -G \\ -Q & -A^* \end{bmatrix}, \quad Q = Q^*,\, G = G^*$$

is a Hamiltonian matrix, i.e., it satisfies $J\mathcal{H} = -\mathcal{H}^* J$, where $J = \begin{bmatrix} & I \\ -I & \end{bmatrix}$.
(Skew-self-adjoint with respect to the antisymmetric scalar product defined by J.)

Spectral symmetry

If $\mathcal{H}v = \lambda v$, then $(v^* J)\mathcal{H} = (-\overline{\lambda})(v^* J)$: eigenvalues are symmetric wrt the imaginary axis.

A similar relation can be proved for Jordan chains: λ and $-\overline{\lambda}$ have Jordan chains of the same size.

Thus, it is sufficient to prove that $\mathcal{H}$ has no pure imaginary eigenvalues to conclude that they split $n : n$ in LHP:RHP.

Solvability conditions

Theorem

Assume $Q \succeq 0$, $G = BR^{-1}B^* \succeq 0$, and (A, B) stabilizable. Then, $\mathcal{H}$ has no eigenvalues with $\operatorname{Re}\lambda = 0$.

Proof (sketch)

Suppose instead $\mathcal{H}\left[\begin{smallmatrix} z_1 \\ z_2 \end{smallmatrix}\right] = \imath\omega\left[\begin{smallmatrix} z_1 \\ z_2 \end{smallmatrix}\right]$; from
$0 = \operatorname{Re}\left[\begin{smallmatrix} z_2^* & z_1^* \end{smallmatrix}\right]\left[\begin{smallmatrix} A & -G \\ -Q & -A^* \end{smallmatrix}\right]\left[\begin{smallmatrix} z_1 \\ z_2 \end{smallmatrix}\right] = z_2^* G z_2 + z_1^* Q z_1$ follows that $Qz_1 = 0$, $z_2^* B = 0$. But the latter together with $-A^* z_2 = -\imath\omega z_2$ contradicts stabilizability.

Hence, $\mathcal{H}$ has n eigenvalues in the LHP and n associated ones in the RHP: it has exactly one stabilizing subspace (of dimension n).

Form of the invariant subspace

We know now that there exists a (unique) stable invariant subspace

$$\operatorname{Im}\begin{bmatrix} U_1 \\ U_2 \end{bmatrix}, \quad U_1, U_2 \in \mathbb{R}^{n\times n}.$$

We would like to show that U_1 is invertible. Then we can write a different basis for the same space

$$\operatorname{Im}\begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = \operatorname{Im}\begin{bmatrix} U_1 \\ U_2 \end{bmatrix} U_1^{-1} = \operatorname{Im}\begin{bmatrix} I \\ X \end{bmatrix}.$$

Suppose (A, B) stabilizable, $Q \succeq 0$, $G \succeq 0$. Then U_1 is invertible.

Proof: sketch in the next slide.

Nonsingularity of U_1

For any v such that $U_1 v = 0$,

$$-v^* U_2^* G U_2 v = \begin{bmatrix} v^* U_2^* & 0 \end{bmatrix} \mathcal{H} \begin{bmatrix} 0 \\ U_2 v \end{bmatrix} = v^* \begin{bmatrix} U_2^* & -U_1^* \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \mathcal{R} v = 0.$$

implies $B^* U_2 v = 0$ and $G U_2 v = 0$. The first block row of

$$\begin{bmatrix} A & -G \\ -Q & -A^* \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} v = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \mathcal{R} v$$

gives $U_1 \mathcal{R} v = 0 \implies \ker U_1$ is $\mathcal{R}$-invariant. If $\ker U_1$ is nontrivial, we can find v, $\lambda \in LHP$ such that $U_1 v = 0$, $\mathcal{R} v = \lambda v$. Now the second block row gives $-A^* U_2 v = \lambda U_2 v$. This (together with $B^* U_2 v = 0$ from above) contradicts stabilizability.

Symmetry of the solution

By Hamiltonian properties, if $\begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$ spans the stable invariant subspace, $\begin{bmatrix} U_1^* & U_2^* \end{bmatrix} J = \begin{bmatrix} U_2^* & -U_1^* \end{bmatrix}$ spans the left anti-stable invariant subspace.

Left and right invariant subspaces relative to disjoint eigenvectors are orthogonal $\implies$

$$0 = \begin{bmatrix} U_2^* & -U_1^* \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = U_2^* U_1 - U_1^* U_2.$$

Hence also

$$U_1^{-*} U_2^* - U_2 U_1^{-1} = 0 \implies X^* - X = 0.$$

Positive semidefiniteness of the solution

Note that

$$ARE \iff (A - GX)^T X + X(A - GX) + Q + XGX = 0.$$

So X solves the Lyapunov equations

$$\hat{A}^T X + X\hat{A} + \hat{Q} = 0, \quad \hat{A} = A - GX,\ \hat{Q} = Q + XGX.$$

And we know that $\Lambda(\hat{A}) \subset LHP, \hat{Q} \succeq 0 \implies X \succeq 0$.

How to solve Riccati equations

- Newton's method.
- Invariant subspace via unstructured methods (QR).
- Invariant subspace via 'semi-structured' methods (Laub trick).
- Invariant subspace via structured methods (URV).
- Doubling / Sign iteration.