Note Title 2020-05-22

$$\boxed{\begin{array}{c} A^* X + XA + Q - XGX = 0 \\ \Lambda(A - GX) = 0 \end{array}}$$

$\Leftrightarrow$ trovare sott. inv. stabile $\operatorname{Im}\begin{bmatrix} I \\ X \end{bmatrix}$ di $\mathcal{H} = \begin{bmatrix} A & -G \\ -Q & -A^* \end{bmatrix}$

1) Newton (ogni iterazione è un'eq. di Lyapunov

2) Tipo-QR (Schur ordinata, fattorizz. URV (strutturata))

(non preservo struttura)

3) Funzione segno, $X_0 = \mathcal{H}$, $X_{k+1} = \frac{1}{2}\left(X_k + \boxed{X_k^{-1}}\right)$

3b) structured doubling algorithm

+ manipolazioni algebriche:

$Y_k = (I - X_k)^{-1}(I + X_k)$ $\qquad Y_{k+1} = -Y_k^2$

X_k Hamiltoniano $\Rightarrow Y_k$ simplettico $\left(Y_k^T J Y_k = J\right)$

(analogo di: se X antisimmetrica, $(I - X)^{-1}(I + X)$ è ortogonale)

($y = \frac{1-x}{1+x}$ manda l'asse immag. nel cerchio unitario)

$$Y_k = \begin{bmatrix} I_n & G_k \\ 0 & F_k \end{bmatrix}^{-1} \begin{bmatrix} E_k & 0 \\ H_k & I_n \end{bmatrix}$$

Data $\mathcal{H}$, posso calcolare i blocchi E_0, F_0, G_0, H_0

tramite $\begin{bmatrix} E_0 & G_0 \\ H_0 & F_0 \end{bmatrix} = \begin{bmatrix} I - \mathcal{H}_{11} & +\mathcal{H}_{12} \\ -\mathcal{H}_{21} & I + \mathcal{H}_{22} \end{bmatrix}^{-1} \begin{bmatrix} I + \mathcal{H}_{11} & -\mathcal{H}_{12} \\ \mathcal{H}_{21} & I - \mathcal{H}_{22} \end{bmatrix}$

Y_k simplettica $\Rightarrow E_0 = F_0^T \quad G_0 = G_0^T \quad H = H_0^T$

$G \succeq 0,\ Q \succeq 0 \Rightarrow G_0 \succeq 0 \quad H_0 \succeq 0$

$$
\begin{tikzcd}
X_0 \arrow[r, mapsto] \arrow[d, leftrightarrow] & \frac{1}{2}(X_0+X_0^{-1}) = X_1 \arrow[d, leftrightarrow] \\
Y_0 \arrow[r, mapsto] \arrow[d, leftrightarrow] & -Y_0^2 = Y_1 \arrow[d, leftrightarrow] \\
E_0, F_0, G_0, H_0 \arrow[r, mapsto, "?"] & E_1, F_1, G_1, H_1
\end{tikzcd}
$$

Dati E_k, F_k, G_k, H_k, trovare

$$
\begin{bmatrix} 1 & G_{k+1} \\ 0 & F_{k+1} \end{bmatrix}^{-1} \begin{bmatrix} E_{k+1} & 0 \\ H_{k+1} & 1 \end{bmatrix} = -\left(\begin{bmatrix} I & G_k \\ 0 & F_k \end{bmatrix}^{-1} \begin{bmatrix} E_k & 0 \\ H_k & I \end{bmatrix} \right)^2 =
$$

$$
= \begin{bmatrix} I & G_k \\ 0 & F_k \end{bmatrix}^{-1} (-1) \underbrace{\begin{bmatrix} E_k & 0 \\ H_k & I \end{bmatrix} \begin{bmatrix} 1 & G_k \\ 0 & F_k \end{bmatrix}^{-1}}_{= \begin{bmatrix} 1 & \hat{G} \\ 0 & \hat{F} \end{bmatrix}^{-1} \begin{bmatrix} \hat{E} & 0 \\ \hat{H} & 1 \end{bmatrix}} \begin{bmatrix} E_k & 0 \\ H_k & I \end{bmatrix}
$$

$$
\underbrace{\left(\begin{bmatrix} 1 & \hat{G} \\ 0 & \hat{F} \end{bmatrix} \begin{bmatrix} 1 & G_k \\ 0 & F_k \end{bmatrix} \right)^{-1}}_{= \begin{bmatrix} 1 & G_{k+1} \\ 0 & F_{k+1} \end{bmatrix}^{-1}} \underbrace{\left(\begin{bmatrix} \hat{E} & 0 \\ \hat{H} & I \end{bmatrix} \begin{bmatrix} E_k & 0 \\ H_k & I \end{bmatrix} \right)}_{= \begin{bmatrix} E_{k+1} & 0 \\ H_{k+1} & I \end{bmatrix}}
$$

$$
-\begin{bmatrix} E_k & 0 \\ H_k & 1 \end{bmatrix} \begin{bmatrix} 1 & G_k \\ 0 & F_k \end{bmatrix}^{-1} \overset{?}{=} \begin{bmatrix} 1 & \hat{G} \\ 0 & \hat{F} \end{bmatrix}^{-1} \begin{bmatrix} \hat{E} & 0 \\ \hat{H} & 1 \end{bmatrix}
$$

$$
\left[\begin{array}{cc|cc} 1 & \hat{G} & \hat{E} & 0 \\ 0 & \hat{F} & \hat{H} & 1 \end{array}\right] \left[\begin{array}{cc} E_k & 0 \\ H_k & 1 \\ \hline 1 & G_k \\ 0 & F_k \end{array}\right] = 0 \iff \left[\begin{array}{cc|cc} 1 & 0 & \hat{E} & \hat{G} \\ 0 & 1 & \hat{H} & \hat{F} \end{array}\right] \begin{bmatrix} E_k & 0 \\ 0 & F_k \\ 1 & G_k \\ H_k & 1 \end{bmatrix} = 0
$$

$$\Leftrightarrow \begin{bmatrix} \hat{E} & \hat{G} \\ \hat{H} & \hat{F} \end{bmatrix} = -\begin{bmatrix} E_k & 0 \\ 0 & F_k \end{bmatrix} \begin{bmatrix} I & G_k \\ H_k & I \end{bmatrix}^{-1}$$

Come calcolare $\begin{bmatrix} I & G_k \\ H_k & I \end{bmatrix}^{-1}$?

$$\begin{bmatrix} I & G_k \\ H_k & I \end{bmatrix} \begin{bmatrix} I & -G_k \\ -H_k & I \end{bmatrix} = \begin{bmatrix} I-G_kH_k & 0 \\ 0 & I-H_kG_k \end{bmatrix}$$

$$\Updownarrow$$

$$\begin{bmatrix} I & G_k \\ H_k & I \end{bmatrix} \underbrace{\begin{bmatrix} I & -G_k \\ -H_k & I \end{bmatrix} \begin{bmatrix} (I-G_kH_k)^{-1} & 0 \\ 0 & (I-H_kG_k)^{-1} \end{bmatrix}}_{= \begin{bmatrix} I & G_k \\ H_k & I \end{bmatrix}^{-1}} = I$$

Mettendo tutto insieme,

(*) SDA "structured doubling algorithm"

$$\begin{cases} E_{k+1} = -E_k(I-G_kH_k)^{-1}E_k \\ F_{k+1} = -F_k(I-H_kG_k)^{-1}F_k \\ G_{k+1} = G_k + E_k\underline{G_k(I-H_kG_k)^{-1}}F_k \\ H_{k+1} = H_k + F_k\underline{H_k(I-G_kH_k)^{-1}}E_k \end{cases}$$

a (trasposte)

blocchi $n \times n$

Costo = stesso di un passo di it. segno $\frac{1}{2}(X_k + \underline{X_k^{-1}})$

1 inversa $2n \times 2n$

$2(2n)^3$

Osservazioni:

G_k, H_k simmetrici $\Rightarrow$ $G_k(I-H_kG_k)^{-1}$ simmetrico

$$\left[G_k(I-H_kG_k)^{-1}\right]^T = (I-G_kH_k)^{-1}G_k = G_k(I-H_kG_k)^{-1}$$

$$\Updownarrow$$

$$G_k(I-H_kG_k) = (I-G_kH_k)G_k$$

$H_k(I-G_kH_k)^{-1}$ analogo

Quindi, $E_k = F_k^T$, $G_k = G_k^T$, $H_k = H_k^T$ si può dim. per induzione

Se $G_0 \succeq 0$, $H_0 \preceq 0$, allora ad ogni passo vale che

$$0 \preceq G_0 \preceq G_1 \preceq G_2 \preceq \ldots \preceq G_k \preceq \ldots$$

$$0 \succeq H_0 \succeq H_1 \succeq H_2 \succeq \ldots \succeq H_k \succeq \ldots$$

Per mostrarlo, ci basta mostrare che $G_K(I-H_KG_K)^{-1} \succeq 0$

$$\Updownarrow$$

$$\underbrace{G_K}_{\succeq 0} - \overbrace{G_K \underbrace{H_K}_{\preceq 0} G_K}^{\text{congruenza}} = (I-G_KH_K)\, G_K \cancel{(I-H_KG_K)}^{-1}\cancel{(I-H_KG_K)} \succeq 0$$

congruenza

notate anche che $I-H_KG_K$

Fatto di alg. lineare: $\Lambda(AB)=\Lambda(BA)$

autovalori di H_KG_K = autovalori di $(H_KG_K^{\frac{1}{2}})\,G_K^{\frac{1}{2}}$

= autovalori di $G_K^{\frac{1}{2}}H_KG_K^{\frac{1}{2}} \preceq 0$

congruenza

$\Rightarrow \Lambda(H_KG_K)$ sono tutti reali e ≤ 0

$\Rightarrow I-H_KG_K$ ha tutti autovalori reali ≥ 1, in part. è invertibile.

(stesso discorso per $I-G_KH_K$)

Teo: Nell'iterazione $(*)$, (A,B) stabilizzabile (A^T,Q) stabilizzabile

$H_K \to -X$ (soluzione stabilizzante della ARE $A^*X+XA+Q-XGX=0$)

e $G_K \to Y$ (soluzione stabilizzante di $YA^*+AY+G-YQY=0$)

"eq. duale"

$E_K = F_K^T \to 0$

la convergenza è quadratica.

"Dim. non formale":

Stiamo facendo potenze successive di Y_0: $\underset{= Y_1}{-Y_0^2}$, $\underset{= Y_2}{-Y_0^4}$, $\underset{= Y_3}{-Y_0^8}$, ...

Ci sono n autovalori che $\to \infty$, n autoval. che $\to 0$

$$Y_K = \begin{bmatrix} I & G_K \\ 0 & F_K \end{bmatrix}^{-1} \begin{bmatrix} E_K & 0 \\ H_K & I \end{bmatrix} \to \begin{bmatrix} I & G_\infty \\ 0 & F_\infty \end{bmatrix}^{-1} \begin{bmatrix} E_\infty & 0 \\ H_\infty & I \end{bmatrix} = Y_\infty$$

deve avere kernel di dimensione n; è possibile sse $E_\infty = 0$, $\begin{bmatrix} I \\ -H_\infty \end{bmatrix} = \ker Y_\infty$

(sott. inv. stab. di Y_0, $\begin{bmatrix} I \\ X \end{bmatrix}$)

Dim: $X_0 = \mathcal{H}$ $\qquad Y_0 = (I-\mathcal{H})^{-1}(I+\mathcal{H})$

$$\mathcal{H}\begin{bmatrix}I\\X\end{bmatrix} = \begin{bmatrix}I\\X\end{bmatrix}R \qquad R = A - GX \quad \text{(relazione di sottosp. invariante)}$$

Mostriamo che:

$$\underbrace{(I-\mathcal{H})^{-1}(I+\mathcal{H})}_{\substack{\text{stessi autovett. di } \mathcal{H},\\ \text{autoval. mappati } \lambda \mapsto \frac{1+\lambda}{1-\lambda}}}\begin{bmatrix}I\\X\end{bmatrix} = \begin{bmatrix}I\\X\end{bmatrix}(I-R)^{-1}(I+R)$$

$$(I+\mathcal{H})\begin{bmatrix}I\\X\end{bmatrix}(I-R) = (I-\mathcal{H})\begin{bmatrix}I\\X\end{bmatrix}(I+R)$$

$$\Updownarrow$$

$$\cancel{\mathcal{H}\begin{bmatrix}I\\X\end{bmatrix}R} + \mathcal{H}\begin{bmatrix}I\\X\end{bmatrix} - \begin{bmatrix}I\\X\end{bmatrix}R + \cancel{\begin{bmatrix}I\\X\end{bmatrix}} = \cancel{\mathcal{H}\begin{bmatrix}I\\X\end{bmatrix}R} - \mathcal{H}\begin{bmatrix}I\\X\end{bmatrix} + \begin{bmatrix}I\\X\end{bmatrix}R + \cancel{\begin{bmatrix}I\\X\end{bmatrix}}$$

$$\Updownarrow$$

$$2\mathcal{H}\begin{bmatrix}I\\X\end{bmatrix} = 2\begin{bmatrix}I\\X\end{bmatrix}R \quad \text{vera per rel. di sottosp. invariante.}$$

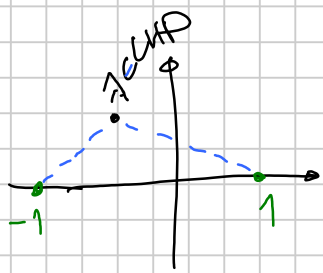
$$Y_0\begin{bmatrix}I\\X\end{bmatrix} = \begin{bmatrix}I\\X\end{bmatrix}\mathcal{T} \qquad \Lambda(\mathcal{T}) \subset \text{cerchio unitario}$$

$$\mathcal{T} = (I-R)^{-1}(I+R) \qquad \Lambda(R) \subset LHP$$

se R ha autoval.: $\lambda_1, \lambda_2, \dots \lambda_n \subset LHP$,

$\mathcal{T}$ ha autoval $\frac{1+\lambda_1}{1-\lambda_1}, \frac{1+\lambda_2}{1-\lambda_2}, \dots, \frac{1+\lambda_n}{1-\lambda_n}$

che stanno tutti nel cerchio unitario



$\lambda \in LHP$ più vicino a -1 che a 1 $\Rightarrow \frac{|1+\lambda|}{|1-\lambda|} < 1 \Rightarrow \frac{1+\lambda}{1-\lambda} \in$ cerchio

$$Y_0\begin{bmatrix}I\\X\end{bmatrix} = \begin{bmatrix}I\\X\end{bmatrix}\mathcal{T} \qquad \Lambda(\mathcal{T}) \subset \text{cerchio unitario} \qquad \rho(\mathcal{T}) < 1$$

$$Y_0^2\begin{bmatrix}I\\X\end{bmatrix} = Y_0\begin{bmatrix}I\\X\end{bmatrix}\mathcal{T} = \begin{bmatrix}I\\X\end{bmatrix}\mathcal{T}^2$$

$$Y_0^3\begin{bmatrix}I\\X\end{bmatrix} = \dots = \begin{bmatrix}I\\X\end{bmatrix}\mathcal{T}^3$$

$$-Y_0 2^k \begin{bmatrix} 1 \\ X \end{bmatrix} = \begin{bmatrix} 1 \\ X \end{bmatrix} \tau^{2^k}$$

$\rightarrow 0$ perché $\rho(\tau) < 1$

$$-Y_0 2^k = \begin{bmatrix} 1 & G_k \\ 0 & F_k \end{bmatrix}^{-1} \begin{bmatrix} E_k & 0 \\ H_k & 1 \end{bmatrix}$$

$$\begin{bmatrix} E_k & 0 \\ H_k & 1 \end{bmatrix} \begin{bmatrix} 1 \\ X \end{bmatrix} = \begin{bmatrix} 1 & G_k \\ 0 & F_k \end{bmatrix} \begin{bmatrix} 1 \\ X \end{bmatrix} (-\tau)^{2^k}$$

$$\Updownarrow$$

congruenza

simmetrica $\rightarrow$
$$\begin{cases} E_k = (1 + G_k X)(-\tau)^{2^k} \\ H_k + X = F_k X (-\tau)^{2^k} = (-\tau^*)^{2^k}(1 + X G_k) X (-\tau)^{2^k} \succeq 0 \end{cases}$$

$\Lambda(XG) \subset \mathbb{R}^+$

1) $H_k \succeq -X$ $\qquad 0 \succeq H_0 \succeq H_1 \succeq H_2 \succeq \dots \succeq -X$

analogamente, per "dualità" $-G_k \succeq -Y \Rightarrow G_k$ limitata

2) se sapessi che $\{G_k\}$ è limitata, potrei concludere:

2a) $E_k \rightarrow 0$ quadraticamente (come $\rho(\tau)^{2^k}$) $\qquad F_k \rightarrow 0$

2b) $H_k + X \rightarrow 0$ quadraticamente, (come $\rho(\tau)^{2 \cdot 2^k}$) $\qquad -G_k + Y \rightarrow 0$

cioè $H_k \rightarrow -X$

Per dimostrare che G_k è limitata, rifaccio tutto questo discorso da zero sull'equazione $AY + YA^* + G - YQY = 0$

(ottenuta scambiando $A \Leftrightarrow A^*$, $G \Leftrightarrow Q$)

Fatto 1: scambiare $A \Leftrightarrow A^*$, $G \Leftrightarrow Q$ corrisponde a scambiare

$E_0 \Leftrightarrow F_0$, $\quad G_0 \Leftrightarrow -H_0$: infatti,

$$\begin{bmatrix} E_0 & G_0 \\ H_0 & F_0 \end{bmatrix} = \begin{bmatrix} 1-\mathcal{H}_{11} & +\mathcal{H}_{12} \\ -\mathcal{H}_{21} & 1+\mathcal{H}_{22} \end{bmatrix}^{-1} \begin{bmatrix} 1+\mathcal{H}_{11} & -\mathcal{H}_{12} \\ \mathcal{H}_{21} & 1-\mathcal{H}_{22} \end{bmatrix} = \begin{bmatrix} 1-A & -G \\ Q & 1-A^* \end{bmatrix}^{-1} \begin{bmatrix} 1+A & G \\ -Q & 1+A^* \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} E_0 & G_0 \\ H_0 & F_0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1-A & -G \\ Q & 1-A^* \end{bmatrix}^{-1} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1+A & G \\ -Q & 1+A^* \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$\shortparallel$

$$\begin{bmatrix} H_0 & F_0 \\ -E_0 & -G_0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1-A & -G \\ Q & 1-A^* \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right)^{-1} \left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1+A & G \\ -Q & 1+A^* \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right) =$$

$$= \left(\begin{bmatrix} Q & 1-A^* \\ -(1-A)^* & G \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right)^{-1} \left(\begin{bmatrix} -Q & 1+A^* \\ -1-A & -G \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right) =$$

$$= \begin{bmatrix} -1-A^* & Q \\ -G & -(1-A^*) \end{bmatrix}^{-1} \begin{bmatrix} 1+A^* & Q \\ -G & 1+A \end{bmatrix}$$

$$\begin{bmatrix} H_0 & F_0 \\ -E_0 & -G_0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} F_0 & -H_0 \\ -G_0 & E_0 \end{bmatrix}$$