$$\|u\|_2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |u(t)|^2 \, dt$$

10 dic 2020

$u \in L^2(-\pi,\pi)$

$$\|u\|_2^2 = \|u - S_n u\|_2^2 + \sum_{|k|\le n} |\hat{u}_k|^2 \quad (*)$$

$e_k(x) = e^{ikx}$

$$S_n u = \sum_{|k|\le n} \langle u, e_k\rangle e_k$$

$$\langle u, e_k\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(t)\, e^{-ikt}\, dt$$

Prop (Q): Se $u \in C_{2\pi}(\mathbb{R},\mathbb{C})$

$S_n u \longrightarrow u$ per $\|\cdot\|_2$

$$\|u\|_2^2 = \sum_{k\in\mathbb{Z}} |\hat{u}_k|^2 \qquad \text{UGUAGLIANZA DI PARSEVAL}$$

OSS: L'uguaglianza deriva da $(*)$ osservando che $\sum_{|k|\le n} |\hat{u}_k|^2 = \|S_n u\|_2^2$

$$(*) \Rightarrow \sum_{|k|\le n} |\hat{u}_k|^2 = \|u\|_2^2 - \|u - S_n u\|_2^2 \xrightarrow[n\to\infty]{} \|u\|_2^2$$

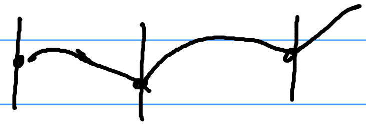
$$\sum_{k\in\mathbb{Z}} |\hat{u}_k|^2 = \|u\|_2^2$$

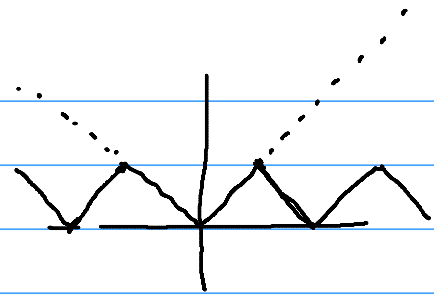
FATTO: Lo stesso risultato vale per $u \in L^2(-\pi,\pi)$

Prop (U) Se $u \in C^1_{2\pi}(\mathbb{R},\mathbb{C})$ allora la serie

$\sum_{k\in\mathbb{Z}} \hat{u}_k e^{ikx}$ è totalm. conv. a u

OSS/Esercizio: Lo stesso risultato vale se u è C^1 a tratti





<u>Esempio</u>: $u(x) = |x| \quad x \in [-\pi, \pi]$

La estendo per periodicità ad R

$u \in C_{2\pi}(\mathbb{R}, \mathbb{R})$

Se $k > 0$ $\quad \hat{u}_k e^{ikx} + \hat{u}_{-k} e^{-ikx} = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(t) \left[e^{-ikt} e^{ikx} + e^{ikt} e^{-ikx} \right] dt$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} u(t) \left[e^{ik(x-t)} + e^{-ik(x-t)} \right] dt$$

$$= \frac{1}{\cancel{2}\pi} \int_{-\pi}^{\pi} u(t)\, \cancel{2} \cos(k(x-t))\, dt$$

$\cos(\alpha - \beta) = \cos\alpha \cos\beta + \sin\alpha \sin\beta$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} u(t) \cos kt\, dt\, \cos kx + \frac{1}{\pi} \int_{-\pi}^{\pi} u(t) \sin(kt)\, dt\, \sin(kx)$$

$$\sum_{k \in \mathbb{Z}} \hat{u}_k e^{ikx} = \hat{u}_0 + \sum_{k \in \mathbb{Z}^*} \hat{u}_k e^{ikx} = \frac{a_0}{2} + \sum_{k=1}^{+\infty} a_k \cos(kx) + b_k \sin(kx)$$

$$a_k \doteq \frac{1}{\pi} \int_{-\pi}^{\pi} u(t) \cos kt\, dt \qquad k \geq 0$$

$$\hat{u}_0 = \frac{1}{2\pi} \int |u(t)|\, dt = \frac{a_0}{2}$$

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} u(t) \sin(kt)\, dt \qquad k \geq 1$$

$\boxed{\text{Sviluppo di Fourier in seni e coseni}}$

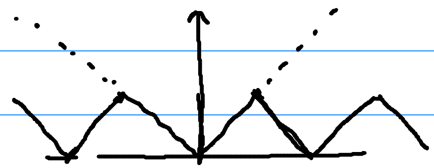
Oss/Esercizio: Se $u : \mathbb{R} \to \mathbb{R}$ 2π-periodica ed L^1

$$u \text{ pari} \iff b_k = 0 \quad \forall k$$

$$u \text{ dispari} \iff a_k = 0 \quad \forall k$$

— o —

Esempio: $u(x) = |x| \quad x \in [-\pi, \pi]$



La estendo per periodicità ad $\mathbb{R}$

$u \in C_r(\mathbb{R}, \mathbb{R})$

u è pari $\Longrightarrow b_k = 0 \quad \forall k \geq 1$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} |t|\, dt = \pi$$

parità

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} |t| \cos(kt)\,dt = \frac{2}{\pi}\int_{0}^{\pi} t\cos(kt)\,dt = \frac{2}{\pi k}\left\{ \cancelto{0}{\left[t \sin(kt)\right]_0^{\pi}} - \int_0^{\pi} \sin kt\, dt \right\}$$

$$= \frac{2}{\pi k^2}\left[-\cos kt\right]_0^{\pi} = \begin{cases} 0 & \text{se } k \text{ pari} \\ -\dfrac{4}{\pi k^2} & \text{se } k \text{ dispari} \end{cases}$$

$$\text{PROP}(U) \Longrightarrow \quad u(x) = \frac{\pi}{2} - \sum_{k \text{ dispari}} \frac{4}{\pi k^2} \cos(kx)$$

(la convergenza è uniforme)

$$0 = u(0) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{k \text{ disp}} \frac{1}{k^2} \Longrightarrow \sum_{k \text{ disp}} \frac{1}{k^2} = \frac{\pi^2}{8}$$

$$\sum_{h=1}^{+\infty} \frac{1}{(2h)^2} = \frac{1}{4}\sum_{h=1}^{+\infty}\frac{1}{h^2} = \frac{1}{4} S$$

$$S = \sum_{k=1}^{+\infty} \frac{1}{k^2} = \underbrace{\sum_{k \text{ disp.}} \frac{1}{k^2}}_{\pi^2/8} + \sum_{k \text{ pari}} \frac{1}{k^2} = \frac{\pi^2}{8} + \frac{1}{4} S$$

$$4S = \frac{\pi^2}{2} + S \implies 3S = \frac{\pi^2}{2} \implies S = \frac{\pi^2}{6}$$

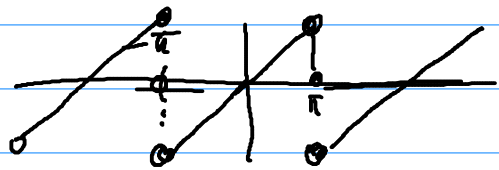
— o —

Esercizio: Calcolare lo sviluppo di Fourier reale delle funzioni

$$\boxed{u_1(x) = x \quad \text{su } [-\pi, \pi]}$$

$$u_2(x) = \chi_{[a,b]}(x) \qquad -\pi \le a < b \le \pi$$

u_1 è continua su $(-\pi, \pi)$ ma la sua estensione periodica a $\mathbb{R}$ è solo regolare a tratti



Prop (P) Se f è 2π-periodica e regolare a tratti allora

$$\forall x_0 \in \mathbb{R} \qquad (S_n f)(x_0) \longrightarrow \frac{1}{2}\left[f(x_0^+) + f(x_0^-) \right]$$

$$f(x_0^{\pm}) = \lim_{x \to x_0^{\pm}} f(x)$$

Def: $\exists\, x_1 \dots x_n$ t.c. $-\pi \le x_1 < x_2 < \dots < x_n \le \pi$ $\quad f|_{(x_i, x_{i+1})}$ si estende ad una funzione C^1

nei punti x_i f può avere disc.

$$(S_n f)(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\, D_n(x-t)\, dt$$

Con $D_n(x) = \sum_{|k|\le n} e^{ikx} = \frac{\sin\left((n+\frac{1}{2})x\right)}{\sin(x/2)}$ se $x \neq 0$

$(D_n(0) = 2n+1)$

D_n è PARI

nucleo di Dirichlet

$t = x + s$

se $\varphi : \mathbb{R} \to \mathbb{R}$ $\varphi \in L^1$ T-periodica $\int_x^{x+T} \varphi(s)\, ds$ non dipende da x

$$= \frac{1}{2\pi}\int_{-\pi-x}^{\pi-x} f(x+s)\, D_n(+s)\, ds$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x+s)\, D_n(s)\, ds$$

$$= \frac{1}{2}\Bigg[\underbrace{\frac{1}{\pi}\int_{-\pi}^{0} f(x+s)\, D_n(s)\, ds}_{\xrightarrow{n\to\infty} f(x^-)} + \underbrace{\frac{1}{\pi}\int_{0}^{\pi} f(x+s)\, D_n(s)\, ds}_{(\star)\ \xrightarrow{n\to\infty} f(x^+)}\Bigg]$$

verifico (✪) (l'altro è analogo.

x fissato.

$$\left| f(x^+) - \frac{1}{\pi}\int_0^{\pi} f(x+s)\, D_n(s)\, ds \right| =$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} D_n(s)\, ds = 1$$

$$\frac{1}{\pi}\int_0^{\pi} D_n(s)\, ds = 1$$

(Perché D_n è pari)

$$= \left| \frac{1}{\pi}\int_0^{\pi} \left[f(x^+) - f(x+s)\right] D_n(s)\, ds \right|$$

$$= \left| \frac{1}{\pi}\int_0^{\pi} \underbrace{\frac{\left[f(x^+) - f(x+s)\right]}{s}} \cdot \frac{s}{\sin(s/2)} \cdot \boxed{\sin\left(\left(n+\tfrac{1}{2}\right)s\right)}\, ds \right|$$

$$D_n(s) = \frac{\sin\left(\left(n+\frac{1}{2}\right)s\right)}{\sin(s/2)}$$

$\frac{s}{\sin(s/2)} \to$ funzione est. per continuità con estensione C'

$\underbrace{} \to$ funzione C' su $[0,\pi] \setminus \{x_1 - x,\ x_2 - x, \dots, x_n - x,\ 0\}$ (mod 2π)

Applico il lemma di Riemann a tutti gli intervalli (x_i, x_{i+1})

$\lambda = n + 1/2$

$$\int_a^b \varphi(t)\, \boxed{\sin(\lambda t)}\, dt \longrightarrow 0 \qquad \lambda \to \infty$$

se $\varphi \in C([a,b]) \cap C'((a,b))$

da cui la tesi ✱

OSS: $\displaystyle\lim_{s\to 0^+} \frac{f(x^+) - f(x+s)}{s} = f'(x^+)$

LEMMA [Fejér] (F) Se $u \in C_{2\pi}(\mathbb{R}, \mathbb{C})$ allora

$$\frac{1}{n} \sum_{\ell=0}^{n-1} S_\ell u \longrightarrow u \quad \text{uniformemente}$$

COR: $\mathrm{span}_{\mathbb{C}} \{ e_k : k \in \mathbb{Z} \}$ è denso in $C_{2\pi}$ per $\| \cdot \|_\infty$

Dim $\frac{1}{n} \sum_{\ell=0}^{n-1} S_\ell u \in V_n$

$$\forall \varepsilon > 0 \quad \exists n : \quad \Big\| \underbrace{\frac{1}{n} \sum_{\ell=0}^{n-1} S_\ell u}_{\in V_n} - u \Big\|_\infty < \varepsilon$$

OSS: $$\frac{1}{n} \sum_{\ell=0}^{n-1} (S_\ell u)(x) = \frac{1}{n} \sum_{\ell=0}^{n-1} \frac{1}{2\pi} \int_{-\pi}^{\pi} u(t) D_\ell(x-t)\, dt$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} u(t) F_n(x-t)\, dt$$

con $F_n(x) \doteq \frac{1}{n} \sum_{\ell=0}^{n-1} D_n(x)$

$\underline{\text{LEMMA}}$ (K) $\quad F_n(x) = \frac{1}{n}\left[\frac{\sin\frac{nx}{2}}{\sin\frac{x}{2}}\right]^2$ per $x \neq 0 \mod 2\pi$

- $F_n(x) \geq 0$
- $\frac{1}{2\pi}\int_{-\pi}^{\pi} F_n(t)\,dt = 1$
- $\therefore\ \forall \delta > 0 \quad \sup_{\delta \leq |x| \leq \pi} F_n(x) \xrightarrow[n\to\infty]{} 0$))

$\underline{\text{Dim}}$ $(x \neq 0)$: $\quad D_n(x) = \sum_{|k|\leq n} e^{ikx} = [\ldots] = \frac{e^{i(n+1)x} - e^{-inx}}{e^{ix} - 1}$

$$(e^{ix}-1)^2 n F_n(x) = (e^{ix}-1)^2 \sum_{\ell=0}^{n-1} D_\ell(x) = (e^{ix}-1)^2 \sum_{\ell=0}^{n-1} \frac{e^{i(\ell+1)x} - e^{-i\ell x}}{e^{ix}-1}$$

$$= \sum_{\ell=0}^{n-1} \left(e^{i(\ell+1)x} - e^{-i\ell x}\right)\left(e^{ix} - 1\right)$$

$$= \sum_{\ell=0}^{n-1} \left(\underbrace{e^{i(\ell+2)x} - e^{i(\ell+1)x}} \underbrace{- e^{-i(\ell-1)x} + e^{-i\ell x}}\right)$$

$$= e^{i(n+1)x} - e^{+ix} - e^{ix} + e^{-i(n-1)x}$$

$$= e^{ix}\left[e^{inx} - 2 + e^{-inx}\right]$$

$$\boxed{(e^{ix}-1)^2 n\, F_n(x) = \left(e^{i x/2}\right)^2 \left[e^{i\frac{nx}{2}} - e^{-i\frac{nx}{2}}\right]^2}$$

$$\left(e^{ix/2} - e^{-ix/2}\right)^2 n\, F_n(x) = \left[e^{i\frac{nx}{2}} - e^{-i\frac{nx}{2}}\right]^2$$

$$n\, F_n(x) = \frac{\left[\cancel{2i}\,\sin\left(\frac{nx}{2}\right)\right]^2}{\left[\cancel{2i}\,\sin\left(\frac{x}{2}\right)\right]^2}$$

Verificare la proprietà (⁘) per esercizio.

—o—

Dim Lemma (F)

$$\Delta_n(x) \doteq u(x) - \frac{1}{n}\sum_{\ell=0}^{n-1} (S_\ell u)(x) \longrightarrow 0 \quad \text{uniformemente in } x$$

$$|\Delta_n(x)| = \left| u(x) - \frac{1}{2\pi}\int_{-\pi}^{\pi} u(t)\, F_n(x-t)\, dt \right|$$

$$\bullet\bullet \to \quad = \left| \frac{1}{2\pi}\int_{-\pi}^{\pi} [u(x) - u(t)]\, F_n(x-t)\, dt \right|$$

$$(\bullet) \quad \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |u(x) - u(t)|\, F_n(x-t)\, dt \qquad s = t - x$$

$$= \frac{1}{2\pi} \int_{-\pi \cancel{-x}}^{\pi \cancel{-x}} |u(x) - u(x+s)| \, F_n(s) \, dt$$

$$\forall \varepsilon > 0 \quad \exists \delta > 0 : \quad |u(x) - u(x+s)| < \varepsilon \quad \forall |s| < \delta$$

HEINE - CANTOR
(unif. cont. di u)

$$\leq \underbrace{\frac{1}{2\pi} \int_{|s| \leq \delta} \varepsilon \cdot F_n(s) \, ds}_{} + \frac{1}{2\pi} \int_{\delta \leq |s| \leq \pi} 2 \|u\|_\infty F_n(s)$$

$$\leq \frac{\varepsilon}{2\pi} + \frac{2\|u\|_\infty}{\cancel{(2\pi)}} \cancel{(2\pi)} \underbrace{\sup_{\delta \leq |s| \leq \pi} F_n(s)}_{\xrightarrow{n \to \infty} 0}$$

$\exists \bar{n} : \forall n \geq \bar{n}$ il secondo addendo è $< \dfrac{\varepsilon}{2\pi}$

$$< \frac{\varepsilon}{\pi} \qquad \forall n \geq \bar{n}$$

$\nwarrow$ stima che non dipende da x