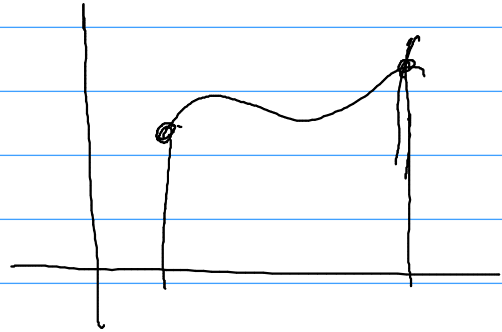
$\boxed{25 \text{ feb } 2021}$

$$\gamma : [a,b] \longrightarrow \mathbb{R}^n$$
$$t \longmapsto \gamma(t)$$

$$\ell(\gamma) = \int_a^b |\gamma'(t)|\, dt$$

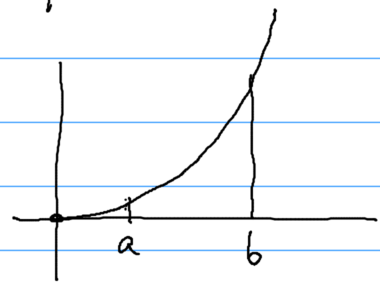
$\mathbb{R}^2$



$$\gamma(t) = (t, \; f(t))$$

$$\int_a^b \sqrt{1 + f'(t)^2}\, dt$$

$f(x) = x^2$



$$\int_a^b \sqrt{1 + 4t^2}\, dt$$

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Curve "espresse" in coordinate polari (curve piane)

$$t \longmapsto \rho(t), \; \theta(t)$$

$$x(t) = \rho(t) \cos\theta(t)$$
$$y(t) = \rho(t) \sin\theta(t)$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \rho' \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} + \rho\theta' \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$

$$x' = \rho' \cos\theta - \rho \sin\theta \; \theta'$$
$$y' = \rho' \sin\theta + \rho \cos\theta \; \theta'$$

$$|\gamma'| = \sqrt{\rho'^2 + \rho^2\theta'^2}$$

$$\ell(\gamma) = \int_a^b \sqrt{\rho'^2 + \rho^2\theta'^2}\, dt$$

$$\boxed{\text{se } \theta(t) = t \qquad \ell(\gamma) = \int_a^b \sqrt{\rho'^2 + \rho^2}\, dt}$$

Esempio: cardioide $\rho = 2A(1+\cos\theta)$ $A>0$

formula in coord. polari $-\pi \le \theta \le \pi$ esercizio: provare a disegnarla

è di classe C^1 per $\theta \in (-\pi, \pi)$

è una curva chiusa, è semplice

$$\rho'(\theta) = -2A\sin\theta$$

$$\ell(\gamma) = 2A\int_{-\pi}^{\pi} \sqrt{\sin^2\theta + (1+\cos\theta)^2}\, d\theta = 2A\int_{-\pi}^{\pi} \sqrt{2(1+\cos\theta)}\, d\theta$$

$$= 8A\int_0^{\pi} \cos\left(\frac{\theta}{2}\right) d\theta = 8A\left[2\sin\frac{\theta}{2}\right]_0^{\pi} \qquad 2(1+\cos\theta) = 4\cos^2\left(\frac{\theta}{2}\right)$$

$$= 16A$$

$$\boxed{1+\cos(2x) = 2\cos^2 x}$$

$$\gamma : [a,b] \longrightarrow \mathbb{R}^n$$

__Prop__: Se γ è una curva C^1 (a tratti) allora ammette una parametrizzazione per lunghezza d'arco

$$\begin{tikzcd}
\gamma : [a,b] \arrow[r, "\gamma"] \arrow[d, "s=s(t)"] & \mathbb{R}^n \\
{[0, \ell(\gamma)]} \arrow[ur, "\tilde\gamma"'] &
\end{tikzcd}$$

$$s(t) := \int_a^t |\dot\gamma(\tau)|\, d\tau \qquad s'(t) = |\gamma'(t)|$$

Parametrizzazione per lunghezza d'arco s

$$\gamma(t) = \tilde\gamma(s(t))$$

$$\gamma'(t) = \tilde\gamma'(s(t)) \cdot s'(t)$$

$$\tilde\gamma'(s(t)) = \frac{\gamma'(t)}{|\gamma'(t)|}$$

vettore unitario

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Se $\tilde\gamma$ è curva param. per lungh d'arco $\quad T(s) = \tilde\gamma'(s)$.

__CURVE PIANE__ $\quad \gamma(t) = (x(t), y(t)) \qquad \tilde\gamma(s) = (\tilde x(s), \tilde y(s))$

Riparametrizzata per lungh d'arco con medesimo verso.

$$T(t) = \frac{\gamma'(t)}{|\gamma'(t)|} = \left(\frac{x'}{|\gamma'|}, \frac{y'}{|\gamma'|}\right) \qquad \tilde T(s) = \tilde\gamma'(s) = (\tilde x'(s), \tilde y(s))$$

$$N(t) = \left(\frac{y'}{|\gamma'|}, \frac{-x'}{|\gamma'|}\right) \qquad \tilde N(s) = (\tilde y, -\tilde x)$$

Se derivo $\tilde T$ ottengo un vettore ortog. a $\tilde T$

$\boxed{\text{CURVATURA SCALARE (per curve } C^2)}$ $\Rightarrow \exists\, k = k(s)$ tale che

$$\tilde T'(s) = -\tilde k(s) \tilde N(s)$$

curvatura scalare

Come si scrive la curvatura nelle coord "t"

$$\gamma(t) = \tilde{\gamma}(s(t))$$

$$T(t) = \frac{\gamma'}{|\gamma'|} = \tilde{T}(s(t))$$

$$N(t) = \tilde{N}(s(t))$$

$$\left(\gamma'(t) = \tilde{\gamma}'(s(t))\, s'(t) = \tilde{T}(s(t))\, |\gamma'(t)|\right)$$

$$\tilde{k}(s) = -\tilde{T}'(s) \cdot \tilde{N}(s)$$

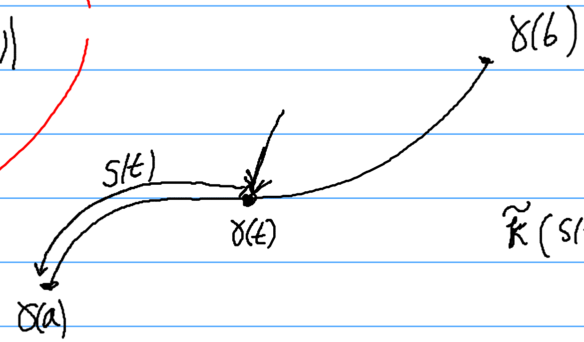
$$T'(t) = \frac{d}{dt}\left(\tilde{T}(s(t))\right) = \tilde{T}'(s(t)) \cdot |\gamma'(t)|$$

$k(t)$

$$\boxed{T'(t) \cdot N(t) = \tilde{T}'(s(t))\, |\gamma'(t)| \cdot \tilde{N}(s(t)) = -\tilde{k}(s(t)) \cdot |\gamma'(t)|}$$

$$T(t) = \left(\frac{x'}{R}, \frac{y'}{R}\right) \qquad R = \sqrt{x'^2 + y'^2} = |\gamma'(t)|$$

↑ dipende da t



$\tilde{k}(s(t))$

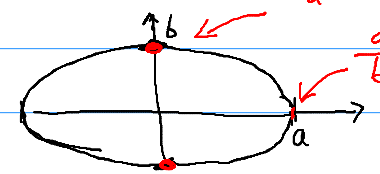
$$T'(t) = \left(\frac{x''R - x'R'}{R^2}, \frac{y''R - y'R'}{R^2}\right)$$

$$N(t) = \left(\frac{y'}{R}, -\frac{x'}{R}\right)$$

$$T' \cdot N = \frac{1}{R^2}\left(x''y' - y''x'\right) \Rightarrow \tilde{k}(s(t)) = \frac{y''x' - x''y'}{(x'^2 + y'^2)^{3/2}} = k(t)$$

<u>Esempio</u>: Ellisse di assi a, b

$$\begin{cases} x(t) = a \cos t \\ y(t) = b \sin t \end{cases}$$



$\frac{b}{a^2}$ curv. minima

$\frac{a}{b^2}$ curv. max

$$k(t) = \frac{ba \sin^2 t + ab \cos^2 t}{(a^2 \sin^2 t + b^2 \cos^2 t)^{3/2}} = \frac{ab}{(a^2 \sin^2 t + b^2 \cos^2 t)^{3/2}}$$

Significato geom.

il modulo della curvatura è il reciproco del raggio di un cerchio osculante

Sia γ curva param. a vel 1 $\quad s_0 \in [a,b]$, fissato $\quad P_0 \in \mathbb{R}^2$

$\gamma \in C^2$

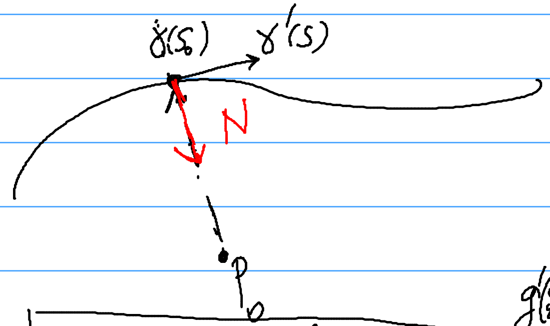
$$g(s) = |\gamma(s) - P_0|^2$$

$$g(s) = g(s_0) + \underline{g'(s_0)}(s-s_0) + \underline{g''(s_0)}\frac{(s-s_0)^2}{2} + o((s-s_0)^2)$$

Q: Come scegliere P_0 in modo che $\boxed{g'(s_0) = g''(s_0) = 0}$?

In questo caso il cerchio

$\{ P \in \mathbb{R}^2 : |P - P_0|^2 = g(s_0) \}$ è osculante alla curva



$$g'(s) = 2(\gamma(s) - P_0) \cdot \gamma'(s)$$

$g'(s_0)=0$

$$\boxed{(\gamma(s_0) - P_0) \cdot \gamma'(s_0) = 0}$$

$$g''(s) = 2\left[|\gamma'|^2 + \underbrace{(\gamma - P_0)}_{\lambda N} \cdot \underbrace{\gamma''(s)}_{-kN} \right]$$

$$\begin{cases} \gamma' = T \\ \gamma'' = T' = -k N \end{cases}$$

$$= 2\left[1 - \lambda(s) k(s) \right]$$

$$\gamma(s_0) - P_0 = \lambda N$$

$$g''(s_0) = 0 \Longrightarrow \lambda(s_0) = \frac{1}{k(s_0)}$$

Con questa scelta di P_0 $\quad B\left(P_0, \frac{1}{k(s_0)}\right)$ è il cerchio osculante