Vettorizzazione e prodotti di Kronecker

Rappresentare mappe lineari matrici $\to$ matrici

$$\boxed{AX - XB} = \boxed{C} \qquad \text{(eq. di Sylvester)}$$

$C, X \in \mathbb{C}^{m\times n}$, $A \in \mathbb{C}^{m\times m}$ $B \in \mathbb{C}^{n\times n}$

$\to$ sistema di mn equaz. lin. in mn incognite

Comb. lineari di elementi di A

$$\mathrm{vec}(X) = \mathrm{vec}\begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & & \vdots \\ x_{m1} & x_{m2} & \cdots & x_{mn} \end{bmatrix} := \begin{bmatrix} x_{11} \\ x_{21} \\ \vdots \\ x_{m1} \\ \hline x_{12} \\ x_{22} \\ \vdots \\ x_{m2} \\ \hline \vdots \\ \hline x_{1n} \\ \vdots \\ x_{mn} \end{bmatrix}$$

$$X \mapsto AXB \qquad \text{matrice } mn \times pq$$

$\mathbin{\rotatebox[origin=c]{90}{$\supset$}}$ $\mathbb{C}^{m\times n}$ $\qquad$ $\mathbb{C}^{p\times q}$

$$\mathrm{vec}(X) \mapsto \mathrm{vec}(AXB)$$

$$(AXB)_{hl} = \sum_i \sum_j A_{hi} X_{ij} B_{jl}$$

$$= \begin{bmatrix} A_{h1}B_{1l} & A_{h2}B_{1l} & \cdots & A_{hm}B_{1l} & | & \cdots \\ A_{h1}B_{2l} & A_{h2}B_{2l} & \cdots & A_{hm}B_{2l} & | & \cdots \\ \vdots \\ A_{h1}B_{nl} & & \cdots & A_{hm}B_{nl} \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{21} \\ \vdots \\ x_{m1} \\ \hline \vdots \quad 2 \\ \hline x_{1n} \\ \vdots \\ x_{mn} \end{bmatrix}$$

$$\operatorname{vec}(AXB) = \begin{bmatrix} b_{11}A & b_{21}A & b_{31}A & \cdots & b_{n1}A \\ \vdots & \vdots & \vdots & & \vdots \\ b_{1q}A & & \cdots & & b_{nq}A \end{bmatrix} \operatorname{vec}(X)$$

$$:= B^T \otimes A$$

<u>Def</u>:

$$X \otimes Y = \begin{bmatrix} x_{11}Y & x_{12}Y & x_{13}Y & \cdots & x_{1m}Y \\ x_{21}Y & x_{22}Y & x_{23}Y & & Y \\ \vdots & & & \ddots & \vdots \\ x_{m1}Y & & \cdots & & x_{mn}Y \end{bmatrix}$$

<u>Proprietà</u>:

- $\operatorname{vec}(AXB) = (B^{\boxed{T}} \otimes A)\operatorname{vec}(X)$ anche se B è complessa
- $(A \otimes B)(C \otimes D) = (AC \otimes BD)$
- $(A \otimes B)^T = A^T \otimes B^T$
- Q_1, Q_2 ortogonali/unitarie $\Rightarrow$ $(Q_1 \otimes Q_2)(Q_1 \otimes Q_2)^*$
  $= (Q_1 \otimes Q_2)(Q_1^* \otimes Q_2^*)$
  $= Q_1Q_1^* \otimes Q_2Q_2^* = I \otimes I$
- triangolare sup $\otimes$ triang. sup $=$ triang. sup.
- diag $\otimes$ diag $=$ diag

$$\nabla \otimes \nabla = \begin{bmatrix} \nabla & \nabla & \nabla & \cdots & \nabla \\ 0 & \nabla & & & \vdots \\ 0 & 0 & \ddots & & \\ \vdots & & & & \\ 0 & \cdots & & 0 & \nabla \end{bmatrix}$$

- se $A = U_1S_1V_1^*$ $\quad B = U_2S_2V_2^*$ sono SVD, anche

$$A \otimes B = \underbrace{(U_1 \otimes U_2)}_{\text{unit.}}\underbrace{(S_1 \otimes S_2)}_{\text{diag.}}\underbrace{(V_1 \otimes V_2)^*}_{\text{unit.}}$$ lo è.

- $\|A \otimes B\| = \|A\| \cdot \|B\|$

norma-2

$$\begin{bmatrix} \sigma_{11} S_2 & & & \\ & \sigma_{21} S & & \\ & & \sigma_{31} S & \\ & & & \ddots \\ & & & & \sigma_{n1} S \end{bmatrix}$$

Eq. Sylvester $m > n$ $A \in \mathbb{C}^{m \times m}$, $B \in \mathbb{C}^{n \times n}$, $C, X \in \mathbb{C}^{m \times n}$

$$\boxed{A}\,\boxed{X} - \boxed{X}\,\boxed{B} = \boxed{C} \qquad (S)$$

Teo: (S) ha una e una sola soluzione sse $\Lambda(A) \cap \Lambda(B) = \emptyset$

Dim: è un sist. di mn eq. lineari in mn incognite

$$\mathrm{vec}(AXI_n) = (I_n \otimes A)\,\mathrm{vec}\,X$$

$$\mathrm{vec}(I_m XB) = (B^T \otimes I_m)\,\mathrm{vec}\,X$$

$$(S) \Longleftrightarrow \underbrace{(I_n \otimes A - B^T \otimes I_m)}\,\mathrm{vec}\,X = \mathrm{vec}\,C$$

(S) ha una sol. $\Leftrightarrow$ $I \otimes A - B^T \otimes I \in \mathbb{C}^{mn \times mn}$ è non singolare

Prendo decomposizione di Schur $A = Q_A T_A Q_A^*$ Q_B, Q_A unit, T_B, T_A tr. sup.

$$B^T = Q_B T_B Q_B^*$$

$$I \otimes A - B^T \otimes I = \underbrace{(Q_B \otimes Q_A)}_{\text{unitaria}} \underbrace{(I \otimes T_A - T_B \otimes I)}_{\text{triang. sup.}} \underbrace{(Q_B^* \otimes Q_A^*)}_{\text{la sua trasp-coniugata}}$$

decompos. di Schur

Autovalori di $I \otimes A - B^T \otimes I$ = elem. diagonali di $I \otimes T_A - T_B \otimes I$

$(\lambda_1, \dots \lambda_m) = \Lambda(A)$ $(\mu_1, \dots \mu_n) = \Lambda(B)$

$$\operatorname{diag}\left(\begin{bmatrix}\lambda_1 & & & & & & & & & \ast \\ & \lambda_2 & & & & & & & & \\ & & \ddots & & & & & & & \\ & & & \lambda_m & & & & & & \\ & & & & \lambda_1 & & & & & \\ & & & & & \ddots & & & & \\ & & & & & & \lambda_m & & & \\ & & & & & & & \lambda_1 & & \\ & & & & & & & & \ddots & \\ 0 & & & & & & & & & \lambda_m\end{bmatrix} - \begin{bmatrix}\mu_1 & & & & & & & & & \ast \\ & \mu_1 & & & & & & & & \\ & & \ddots & & & & & & & \\ & & & \mu_1 & & & & & & \\ & & & & \mu_2 & & & & & \\ & & & & & \mu_2 & & & & \\ & & & & & & \ddots & & & \\ & & & & & & & \mu_2 & & \\ & & & & & & & & \mu_n \ddots & \\ 0 & & & & & & & & & \mu_n\end{bmatrix}\right)$$

$= \lambda_i - \mu_j$ per $i=1,\dots m$, $j=1,\dots n$. (differenze tra gli autoval. di A e quelli di B)

$I \otimes A - B^T \otimes I$ invertibile $\Leftrightarrow$ non ha autoval. nulli $\Leftrightarrow \lambda_i - \mu_j \neq 0 \quad \forall i,j$

$\Leftrightarrow \Lambda(A) \cap \Lambda(B) = \emptyset$

Come risolvo numericamente? Vettorizzando $\to O((mn)^3)$ troppo!

Bartels-Stewart: $O(m^3+n^3)$

Idea: inverto $\left((Q_B \otimes Q_A)(I \otimes T_A - T_B \otimes I)(Q_B^* \otimes Q_A^*)\right)^{-1} \operatorname{vec}(C)$

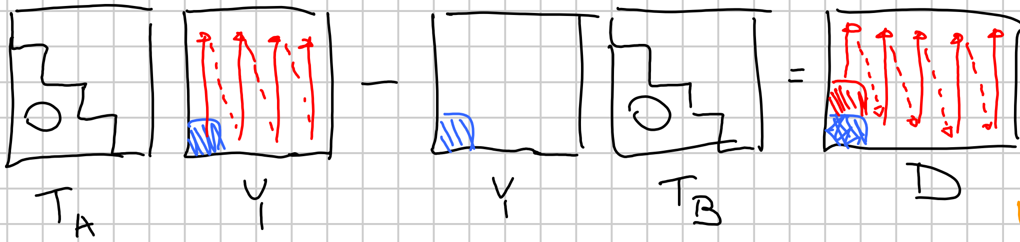
$$= (Q_B \otimes Q_A)\underbrace{(I \otimes T_A - T_B \otimes I)^{-1}}_{\text{sostituzione all'indietro } O(m^3+n^3)} \underbrace{(Q_B^* \otimes Q_A^*) \cdot \operatorname{vec} C}_{\operatorname{vec}(\underbrace{Q_A^* C \overline{Q_B}}_{O(m^3+n^3)})}.$$

$AX - XB = C \qquad A = Q_A T_A Q_A^* \qquad O(m^3) \approx 30m^3$

$B = \hat{Q}_B \hat{T}_B \hat{Q}_B^* \qquad O(n^3) \approx 30n^3$

$$\cancel{Q_A^*} Q_A T_A \underbrace{Q_A^* X \hat{Q}_B}_{=:Y} - \underbrace{Q_A^* X \hat{Q}_B}_{=:Y} \hat{T}_B \cancel{\hat{Q}_B^* \hat{Q}_B} = \underbrace{\boxed{Q_A^* C \hat{Q}_B}}_{=:D} \quad O(m^3+n^3)$$

$$T_A Y - Y T_B = D$$

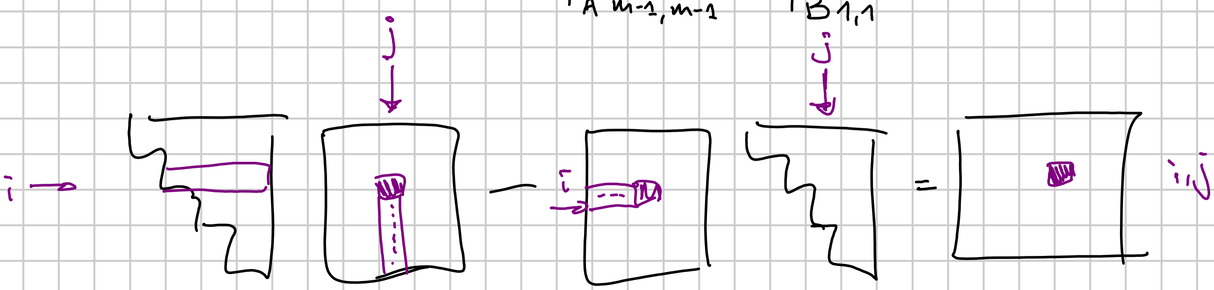


un entrate calcolate
ognuna costo lin. $\Rightarrow$ $O(m^3+n^3)$

in pos. $(m,1)$: $\quad (T_A)_{mm} Y_{m1} - Y_{m1}(T_B)_{11} = D_{m1} \Rightarrow Y_{m1} = \dfrac{D_{m1}}{(T_A)_{mm}-(T_B)_{11}}$.

in pos. $(m-1,1)$ $\quad (T_A)_{m-1,m-1} Y_{m-1,1} + \underbrace{(T_A)_{m-1,m} Y_{m1}}_{\text{nota}} - Y_{m-1,1}(T_B)_{11} = \underline{D_{m-1,1}}$

$$\Rightarrow Y_{m-1,1} = \frac{D_{m-1,1} - (T_A)_{m-1,m} Y_{m1}}{T_{A\,m-1,m-1} - T_{B\,1,1}}$$



$$(T_A)_{ii} Y_{ij} + (\text{roba sotto } Y_{ij}) - \left(Y_{ij}(T_B)_{jj} + (\text{roba a sx di } Y_{ij}) \right) = D_{ij}$$

$$Y_{ij} = \frac{(\text{roba})}{T_{A\,ii} - T_{B\,jj}}$$

$O(m^3+n^3)$

In questo modo calcolo $Y = Q_A^* X Q_B \Rightarrow \underline{X = Q_A Y Q_B^*}$

In totale, $O(m^3+n^3)$

Note

$\Rightarrow$ funziona tutto anche con forme di Schur reali

$\rightarrow$ funziona con tecniche simili per $AXB - CXD = E$

$\rightarrow$ <u>non</u> c'è una generalizzazione $AXB - CXD - EXF = G$

La soluzione di sist. triangolare è stabile all'indietro

$\tilde{Y}$ calcolata risolve esattamente $(M+E)\,\mathrm{vec}\,\tilde{Y} = \mathrm{vec}\,D$

$$\|E\| \leq O(mn)\,\|M\|\,u$$

(ogni operazione messa $\Leftrightarrow$ perturbaz. di un'entrata di M)

$\Rightarrow$ $\tilde{X}$ risolve un sist. del tipo $(I\otimes A - B^T\otimes I + \underset{\uparrow}{F})\,\mathrm{vec}(\tilde{X}) = \mathrm{vec}\,(C)$

errore piccolo risp. a $\|I\otimes A - B^T\otimes I\|$

È vero invece che $\tilde{X}$ risolve esattamente $\tilde{A}\tilde{X} - \tilde{X}\tilde{B} = \tilde{C}$? No!

$A, B, C, \tilde{X}$ Voglio i più piccoli $\delta A, \delta B, \delta C$ tali che

$$(A+\delta A)\tilde{X} - \tilde{X}(B+\delta B) = C + \delta C$$

$$\Leftrightarrow \quad \delta A\cdot\tilde{X} - \tilde{X}\cdot\delta B - \delta C = \underbrace{C - A\tilde{X} - \tilde{X}B}_{\to\ 0 \text{ in aritm. esatta}} = G$$

(comunque piccolo nel nostro caso)

$$\Updownarrow$$

$$\underbrace{\left[\tilde{X}^T\otimes I \;\vdots\; I\otimes\tilde{X} \;\vdots\; I\right]}_{\in\,\mathbb{C}^{mn\times 3mn}} \cdot \underbrace{\begin{bmatrix} \mathrm{vec}\,\delta A \\ \hdashline \mathrm{vec}\,\delta B \\ \hdashline \mathrm{vec}\,\delta C \end{bmatrix}}_{\in\,\mathbb{C}^{3mn}} = \underbrace{\mathrm{vec}\,G}_{\in\,\mathbb{C}^{mn}}$$

$$\begin{bmatrix} \mathrm{vec}\,\delta A \\ \mathrm{vec}\,\delta B \\ \mathrm{vec}\,\delta C \end{bmatrix} = \underbrace{\left[\tilde{X}^T\otimes I \quad I\otimes\tilde{X} \quad I\right]^+}_{\text{si può calcolare supponendo } \tilde{X} \text{ diagonale}}\ \mathrm{vec}\,G$$

è grande se X mal cond.