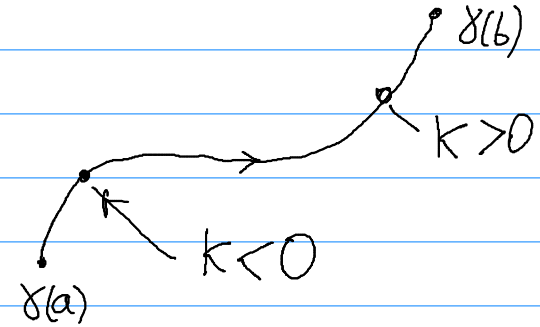
$$\boxed{\text{3 mar 2021}}$$

CURVATURA DI UNA CURVA PIANA

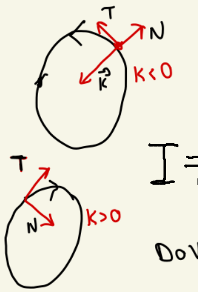
$$\boxed{\tilde{T}' = \pm k \tilde{N}}$$

$\gamma : [a,b] \longrightarrow \mathbb{R}^2$

$$k(t) = \frac{y''x' - x'x''}{(x'^2 + y'^2)^{3/2}}$$



ESERCIZIO: $\gamma : [a,b] \to \mathbb{R}^2$ CURVA REGOLARE, CHIUSA $(\gamma(a) = \gamma(b))$, INIETTIVA, DI CLASSE C^2 $\Rightarrow$

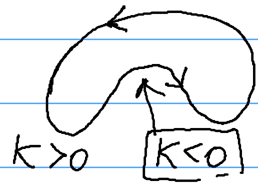


$$I \doteq \int_a^b k(t)\,|\gamma'(t)|\,dt = \pm 2\pi$$

DOVE IL SEGNO DIPENDE DALL'ORIENTAZIONE DELLA CURVA

OSS 1. L'enunciato è vero con la def che abbiamo dato

$\boxed{\text{CURVATURA CON SEGNO}}$

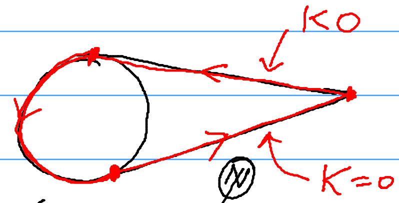


ESERCIZIO: $\gamma : [a,b] \to \mathbb{R}^2$ CURVA REGOLARE, CHIUSA $(\gamma(a) = \gamma(b))$, INIETTIVA, DI CLASSE C^2 $\Rightarrow$

$$\int_a^b k(t)\,|\gamma'(t)|\,dt = \pm 2\pi$$

DOVE IL SEGNO DIPENDE DALL'ORIENTAZIONE DELLA CURVA

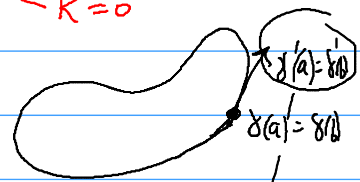
OSS2.



$\gamma(a) = \gamma(b)$ & $\gamma'(a) = \gamma'(b)$

$$I = \int_a^b \frac{y''x' - x''y'}{(x'^2 + y'^2)^{3/2}} (x'^2 + y'^2)^{1/2}\,dt = \int_a^b \frac{y''x' - x''y'}{x'^2 + y'^2}\,dt$$

$\gamma(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$



φ è curva chiusa in $\mathbb{R}^2$

$\varphi(t) = \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix}$ $\qquad \varphi(a) = \varphi(b)$

$\varphi(t) \neq 0$

Scrivo φ in coord polari

$$\boxed{\begin{pmatrix}x'\\ y'\end{pmatrix} = \varphi(t) = \rho(t)\begin{pmatrix}\cos\theta(t)\\ \sin\theta(t)\end{pmatrix}}$$

$$\varphi(a)=\varphi(b) \Longrightarrow \rho(a)=\rho(b)$$
$$\theta(a) = \theta(b) + 2k\pi \quad \text{con } k\in\mathbb{Z}$$

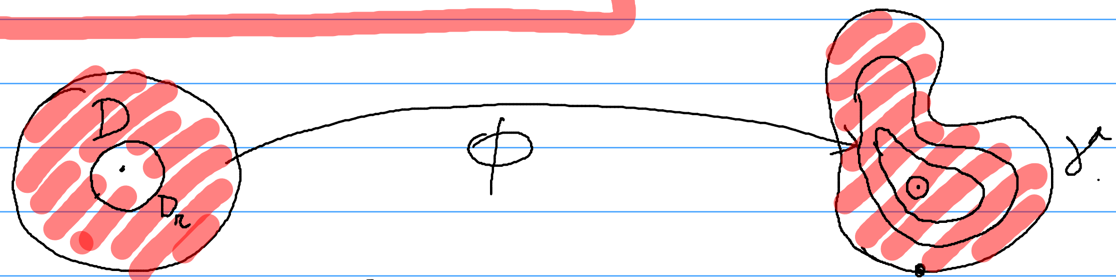
OSS: $I = \int_{\varphi} \omega$ $\qquad \boxed{\omega = \dfrac{x\,dy - y\,dx}{x^2+y^2}}$ $\leftarrow$ Forma $\mathbb{R}^2\setminus\{0\}$ CHIUSA ma non ESATTA

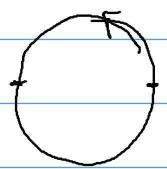
$$\begin{pmatrix}x''\\ y''\end{pmatrix} = \varphi'(t) = \rho'(t)\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix} + \rho(t)\begin{pmatrix}-\sin\theta\\ \cos\theta\end{pmatrix}\theta'(t); \qquad \boxed{\begin{pmatrix}y''\\ -x''\end{pmatrix} = \rho'\begin{pmatrix}s\\ -c\end{pmatrix} + \boxed{\rho\begin{pmatrix}c\\ s\end{pmatrix}\theta'}}$$

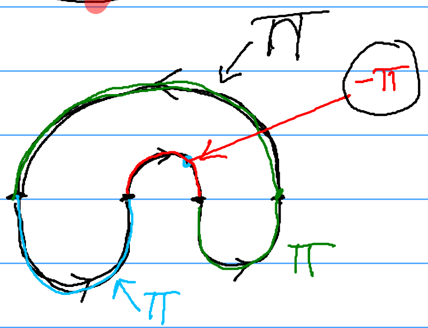
$$N = \begin{pmatrix}x'\\ y'\end{pmatrix}\cdot\begin{pmatrix}y''\\ -x''\end{pmatrix} = \rho^2\theta'$$

$$I = \int_a^b \frac{\cancel{\rho^2}\theta'}{\cancel{\rho^2}}\,dt = \theta(b)-\theta(a) = 2k\pi \quad \text{con } k\in\mathbb{Z}.$$

$\boxed{\text{Curva } \gamma \text{ è iniettiva} \Longrightarrow k = \pm 1}$





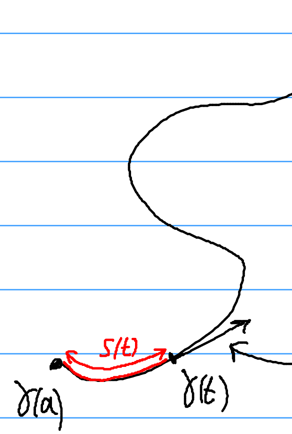


$$\gamma = \Phi(\partial D_1)$$
$$\gamma_r = \Phi(\partial D_r)$$

$(n > 2)$

$\gamma : [a,b] \longrightarrow \mathbb{R}^n$

curva regolare
(derivabile e $\gamma'(t) \neq 0 \quad \forall t$)



versore tangente

$T(t) = \dfrac{\gamma'(t)}{|\gamma'(t)|} = \tilde{T}(s(t))$

$\tilde{\gamma}$ parametrizz. per lungh. d'arco con la stessa orientazione

$s(t) = \int_a^t |\gamma'(\tau)|\, d\tau \qquad L = s(b)$

$\tilde{\gamma} : [0, L] \longrightarrow \mathbb{R}^n \qquad \tilde{\gamma}(s(t)) = \gamma(t) \qquad \tilde{\gamma}(L) = \tilde{\gamma}(s(b)) = \gamma(b)$

$\tilde{T}(s) = \tilde{\gamma}'(s)$

$\tilde{K}(s) = \tilde{T}'(s)$ vettore curvatura $\qquad K(t) = \tilde{K}(s(t))$

$$\frac{d}{dt}\left(T(t)\right) = \frac{d}{dt}\left(\tilde{T}(s(t))\right) = \underbrace{\tilde{T}'(s(t))} \cdot s'(t)$$

$$= \tilde{K}(s(t)) \cdot |\gamma'(t)|$$

$$K(t) = \frac{T'(t)}{|\gamma'(t)|} \qquad\qquad k(t) = |K(t)|$$

$$= [\dots] = \frac{1}{|\gamma'|^2}\left[\gamma'' - (\gamma'' \cdot T)T\right]$$

DEF: γ è BIREGOLARE $\quad K(t) \neq 0 \quad \forall t \qquad N(t) := \dfrac{K(t)}{|K(t)|}$

$K(t) = k(t) \cdot N(t)$

Nel caso di $\mathbb{R}^3$, una curva biregolare individua un sistema di riferimento di versori ortogonali

$\gamma : [a,b] \to \mathbb{R}^3$ param. per lungh. d'arco

$T(s) = \gamma'(s)$

$K(s) = T'(s) = \gamma''(s) = \kappa(s) N(s)$

$B(s) = T(s) \wedge N(s) \leftarrow$ Prodotto vettoriale di $\mathbb{R}^3$

PROD. VETTORIALE $\qquad \mathbb{R}^3 = \text{span}\langle e_1, e_2, e_3 \rangle$

$u, v \in \mathbb{R}^3$

$$u \wedge v = \boxed{\begin{vmatrix} e_1 & e_2 & e_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} e_1 + \begin{vmatrix} u_3 & u_1 \\ v_3 & v_1 \end{vmatrix} e_2 + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} e_3$$

$u \wedge v = \boxed{u \times v}$ ← notazione equivalente

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Proprietà (i) $u \wedge v = -v \wedge u$ antisimmetrico
$u \wedge u = 0$

(ii) $(\lambda u + \mu w) \wedge v = \lambda u \wedge v + \mu w \wedge v$

OSS:
$e_1 \wedge e_2 = e_3$
$e_2 \wedge e_3 = e_1$
$e_3 \wedge e_1 = e_2$

$e_1 \to e_2 \to e_3 \to e_1$

(iii) $u \wedge v = \underline{0} \iff u$ e v sono linearmente dipendenti

$$\begin{pmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{pmatrix} = M$$

($u \wedge v = \underline{0}$ sse tutti i minori 2x2 della matrice sono nulli
$\text{rg}\, M \leq 1 \iff u$ e v lin. dipendenti)

(iv) $$\underbrace{(u \wedge v) \cdot w}_{\text{SCALARE}} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} w_1 + \begin{vmatrix} u_3 & u_1 \\ v_3 & v_1 \end{vmatrix} w_2 + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} w_3 = \begin{vmatrix} w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

conseguenza: $(u \wedge v) \cdot v = 0 = (u \wedge v) \cdot u$

(v) Se u, v sono lin. indipendenti $\text{span}\{u, v, u \wedge v\} = \mathbb{R}^3$

(vi) $(u \wedge v) \wedge w = (u \cdot w)\, v - (v \cdot w)\, u$ $\quad (*)$

Dim: 1. verifico la relazione per versori della base canonica
2. Estendo per linearità $\quad u = e_i, \; v = e_j, \; w = e_k$

1. ∘ se $\{i,j,k\} = \{1,2,3\} \Rightarrow$ entrambi i membri di $(*)$ si annullano $\quad i,j,k \in \{1,2,3\}$

- $(e_i \wedge e_i) \wedge e_k$ $\quad (i=j)$ questo si annulla
e a dx c'è $\quad (e_i \cdot e_k)\, e_i - (e_i \cdot e_k)\, e_i = 0$
- $(e_1 \wedge e_2) \wedge e_2 = e_3 \wedge e_2 = -e_1$
$(e_1 \cdot e_2)\, e_2 - (e_2 \cdot e_2) \cdot e_1 = -e_1$

$e_1 \to e_2 \to e_3 \to e_1$

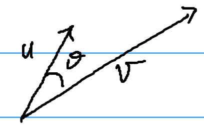
2. $u = \sum u_i e_i$
$v = \sum v_j e_j$
$w = \sum w_k e_k$

$$(u \wedge v) \wedge w = \sum_{ijk} u_i v_j w_k \, (e_i \wedge e_j) \wedge e_k$$

$$= \sum_{ijk} u_i v_j w_k \left[(e_i \cdot e_k)\, e_j - (e_j \cdot e_k)\, e_i \right]$$

$$= \left[\left(\sum_i u_i e_i\right) \cdot \left(\sum_j w_k e_k\right) \sum_j v_j e_j - \left(\sum_j v_j e_j\right) \cdot \left(\sum_k w_k e_k\right) \sum_i u_i e_i \right]$$

OSS: Se u, v sono versori ortogonali $\begin{pmatrix} |u| = |v| = 1 \\ u \cdot v = 0 \end{pmatrix}$



allora $(u \wedge v) \wedge u = v$

$(u \wedge v) \wedge v = -u$

(vii) $$|u \wedge v|^2 = \begin{vmatrix} u \cdot u & u \cdot v \\ u \cdot v & v \cdot v \end{vmatrix} = |u|^2 |v|^2 - (u \cdot v)^2 = |u|^2 |v|^2 \left(1 - \left[\frac{(u \cdot v)}{|u||v|} \right]^2 \right)$$

dimostrazione usando la linearità (x esercizio)

(viii) $t \longmapsto u(t)$
$t \longmapsto v(t)$
$$\frac{d}{dt}\big(u(t) \wedge v(t)\big) = u'(t) \wedge v(t) + u(t) \wedge v'(t)$$

Dim: con taylor $u(t+\varepsilon) = u(t) + \varepsilon u'(t) + o(\varepsilon)$
$v(t+\varepsilon) = v(t) + \varepsilon v'(t) + o(\varepsilon)$ per $\varepsilon \to 0$

$$u(t+\varepsilon) \wedge v(t+\varepsilon) = u(t) \wedge v(t) + \varepsilon \big(u'(t) \wedge v(t) + u(t) \wedge v'(t) \big) + o(\varepsilon)$$

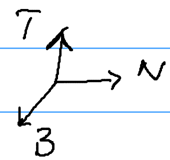
$T(s) = \gamma'(s)$

$T'(s) = \gamma''(s) = k N$

$N' \cdot N = 0$

$B = T \wedge N$

$B'(s) = \cancel{T' \wedge N} + T \wedge N' = \tau N$



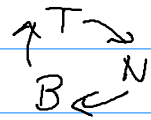
$B \wedge T = (T \wedge N) \wedge T = N$

$$\begin{cases} T' = \kappa N \\ B' = \tau N \\ N' = -(\tau B + \kappa T) \end{cases}$$

$$N' = (B \wedge T)' = B' \wedge T + B \wedge T'$$

$$= \tau N \wedge T + \kappa B \wedge N$$

$$= -\tau B - \kappa T$$



κ = curvatura

τ = "torsione"

<u>Esercizio</u>: Mostrare che se $\gamma(t)$ è parametrizzazione qualunque

$$\kappa(t) = \frac{|\gamma''(t) \wedge \gamma'(t)|}{|\gamma'(t)|^3}$$

formula generale per il calcolo della curvatura