$\boxed{5 \text{ mar } 2021}$

Calcolare curv. usando prodotto vettoriale in $\mathbb{R}^3$ $\quad k(t) = \dfrac{|\gamma'' \wedge \gamma'|}{|\gamma'|^3}$

$$K(t) \wedge T(t) = \frac{1}{|\gamma'|}\left[\frac{\gamma'}{|\gamma'|}\right]' \wedge T$$

$$= \frac{1}{|\gamma'|}\left[\frac{\gamma''}{|\gamma'|} - \gamma' \frac{\frac{d}{dt}|\gamma'|}{|\gamma'|^2}\right] \wedge T = \frac{1}{|\gamma'|} \cdot \frac{\gamma''}{|\gamma'|} \wedge \frac{\gamma'}{|\gamma'|}$$

scalare

$$K(t)\wedge T(t) = k N(t) \wedge T(t) = -kB$$

$$-kB = \frac{1}{|\gamma'|^3}\gamma'' \wedge \gamma' \Rightarrow k = \frac{|\gamma'' \wedge \gamma'|}{|\gamma'|^3}$$

curvatura

Se γ curva param. per lungh d'arco allora i versori $\boxed{T, N, B}$ soddisfano sistema ortonormale orientato positivam.

$$\begin{cases} T' = kN \\ B' = \tau N \\ N' = -kT - \tau B \end{cases}$$

$$\begin{pmatrix} T_1 & N_1 & B_1 \\ T_2 & N_2 & B_2 \\ T_3 & N_3 & B_3 \end{pmatrix}' = \begin{pmatrix} T_1 & N_1 & B_1 \\ T_2 & N_2 & B_2 \\ T_3 & N_3 & B_3 \end{pmatrix} \underbrace{\begin{pmatrix} 0 & -k(t) & 0 \\ k(t) & 0 & \tau(t) \\ 0 & -\tau(t) & 0 \end{pmatrix}}_{A(t)}$$

A antisimmetrica

$$\begin{cases} X' = X \cdot A(t) \\ X(0) = I \end{cases}$$

supponiamo una sol X di questa eq.

Esercizio:

Verificare che una tale soluzione verifica ${}^tX(t)\, X(t) = I \quad \forall t \in \mathbb{R}$

- $\det(X(t)) = 1$

$$(e_1 \wedge e_2)\cdot(e_1\wedge e_3)$$

$(i,j)=(1,2)$

(vii) $$|u\wedge v|^2 = \begin{vmatrix} u\cdot u & u\cdot v \\ u\cdot v & v\cdot v \end{vmatrix}$$

Passo 1 lo verifico per $u=e_i$ $v=e_j$ $\neq$

$u=\sum u_i e_i$

$v=\sum v_j e_j$

$i=j$

$h=k$

$$|u\wedge v|^2 = \Big|\sum_{ij} u_i v_j\, e_i\wedge e_j\Big|^2 = \Big(\sum_{ij} u_i v_j\, e_i\wedge e_j\Big)\cdot\Big(\sum_{h,k} u_h v_k\, e_h\wedge e_k\Big)$$

$$= \sum_{hkij} u_i u_h v_j v_k\,(e_i\wedge e_j)\cdot(e_h\wedge e_k) = \sum_{i\neq j}\Big[(u_i v_j)^2 - u_i v_j u_j v_i\Big]$$

$$= \sum_{1\leq i,j\leq 3}\Big[u_i^2 v_j^2 - u_i v_j u_j v_i\Big] = |u|^2|v|^2 - (u\cdot v)^2$$

TEOREMA di GAUSS GREENE

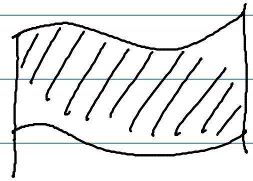
Def: $D \subseteq \mathbb{R}^2$ dominio regolare se

$D = \bigcup_{1 \le i \le n} D_i$ $\quad$ D_i domini normali risp. a x o a y

$int(D_i) \cap int(D_j) = \emptyset \quad$ se $i \neq j$

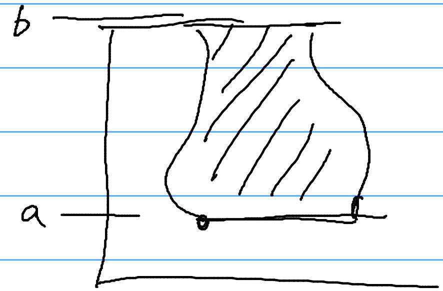
— o —

D è normale risp a x se



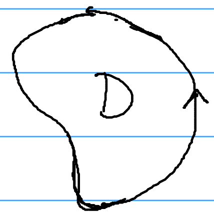
$D = \{(x,y) \in \mathbb{R}^2 : a \le x \le b, \ \alpha(x) \le y \le \beta(x)\}$

con α, β funz. C^1



THM: $\Omega \subseteq \mathbb{R}^2 \qquad P, Q \in C^1(\Omega)$

$D \subseteq \Omega$ dominio regolare. Allora



$$\int_{+\partial D} P\,dx + Q\,dy = \iint_D [\partial_x Q - \partial_y P]\, dx\,dy$$

$$\int_{+\partial D_1} P\,dx + Q\,dy = \iint_{D_1} [\qquad]\, dx\,dy$$

$$\int_{+\partial D_2} P\,dx + Q\,dy = \iint_{D_2} [\qquad]\, dx\,dy$$

— $\qquad$ +

$$\int_{+\partial D} P\,dx + Q\,dy = \iint_{D=D_1\cup D_2} [\partial_x Q - \partial_y P]\,dx\,dy$$

Cor: $D \subseteq \mathbb{R}^2$ dominio regolare allora

$$\text{Area}(D) = \int_{+\partial D} x\,dy = \boxed{-\int_{+\partial D} y\,dx} = \frac{1}{2}\int_{+\partial D} x\,dy - y\,dx$$

$\iint_D 1\,dx\,dy$

$\begin{cases} Q(x,y) = x \\ P = 0 \end{cases}$ $\qquad \begin{cases} P(x,y) = -y \\ Q = 0 \end{cases}$

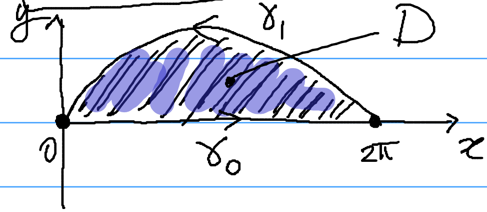
Esempio di applicazione:

D limitato da asse x e curva

$a > 0$

$$\begin{cases} x(t) = (t - \sin t) \\ y(t) = (1 - \cos t) \end{cases} \qquad 0 \le t \le 2\pi$$

$-\gamma_1$



$x(0) = 0$ $\qquad x'(t) \ge 0$

$y(0) = 0$ $\qquad y(t) \ge 0$

$x(2\pi) = 2\pi$

$y(2\pi) = 0$

$$\text{Area } D = \int_{+\partial D} -y\,dx = \int_{\gamma_0} -y\,dx + \int_{\gamma_1} -y\,dx = \int_{-\gamma_1} y\,dx = (*)$$

$y = 0$ su γ_0

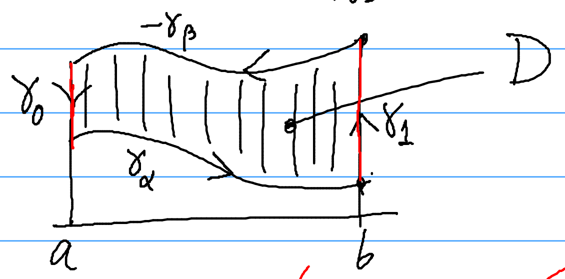
$$(*) = \int_0^{2\pi} (1-\cos t)\cdot(1-\cos t)\,dt = \int_0^{2\pi} (1-\cos t)^2\,dt = \int_0^{2\pi} (1 - 2\cos t + \cos^2 t)\,dt$$

$$= \int_0^{2\pi} (1+\cos^2 t)\,dt = 2\pi + \pi = 3\pi$$

Dimostro direttamente il corollario per domini $\boxed{\text{normali risp a } X}$

$$\text{Area}(D) = -\int_{+\partial D} y\,dx$$

- Dimostro la formula $-y\,dx$ (facile)
- Poi $x\,dy$ (meno facile)



$\gamma_\alpha : t \to \begin{pmatrix} t \\ \alpha(t) \end{pmatrix}$ $\gamma_\alpha : [a,b] \to \mathbb{R}^2$

$\gamma_\beta : t \to \begin{pmatrix} t \\ \beta(t) \end{pmatrix}$ $\gamma_\beta : [a,b] \to \mathbb{R}^2$

$$-\int_{+\partial D} y\,dx = -\left[\cancel{\int_{\gamma_0} y\,dx} + \int_{\gamma_\alpha} y\,dx + \cancel{\int_{\gamma_1} y\,dx} - \int_{\gamma_\beta} y\,dx\right]$$

$$\beta(t)-\alpha(t) = \int_{\alpha(t)}^{\beta(t)} 1\,dy$$

$$= -\left[\int_a^b [\alpha(t) - \beta(t)]\,dt\right] = \int_a^b \left(\int_{\alpha(x)}^{\beta(x)} 1\,dy\right) dx = \iint_D dx\,dy$$

Come faccio a $\iint_D dx\,dy \overset{?}{=} \int_{+\partial D} x\,dy$

Considero $F(x,y) = xy$ $F_x\,dx + F_y\,dy = y\,dx + x\,dy$ forma esatta

$$0 = \int_{+\partial D} F_x\,dx + F_y\,dy = \int_{+\partial D} y\,dx + x\,dy = 0$$

$$-\int_{+\partial D} y\,dx = \text{Area } D$$

$$\int_{+\partial D} x\,dy = \text{Area}(D)$$ $\oplus$

Dimostro Gauss Green (caso generale)

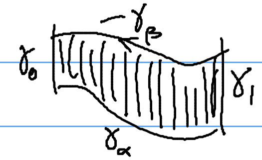
LEMMA: $\boxed{\int_{+\partial D} P\,dx = -\iint_D \partial_y P\,dx\,dy}$ se D normale risp. a x o rispetto a y

$$\int_{+\partial D} Q\,dy = \iint_D \partial_x Q\,dx\,dy$$

— o —

SPG: Dimostro la prima formula per $\boxed{D \text{ normale risp a } x}$ (*)

(" " risp a y) (**)

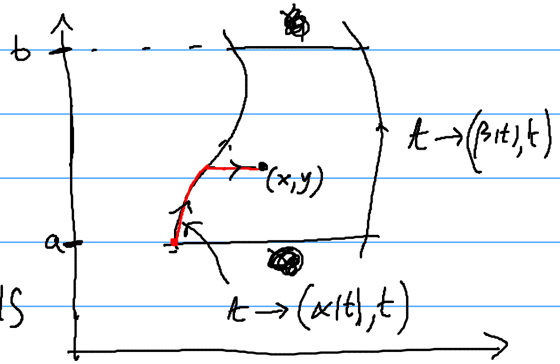


$$\int_{+\partial D} P\,dx = \cancel{\int_{\gamma_0} P\,dx} + \int_{\gamma_\alpha} P\,dx + \cancel{\int_{\gamma_1} P\,dx} - \int_{\gamma_\beta} P\,dx$$

$$= \int_a^b P(x,\alpha(x)) - P(x,\beta(x))\,dx = \int_a^b \left(-\int_{\alpha(x)}^{\beta(x)} \frac{\partial P}{\partial y}(x,y)\,dy\right) dx = -\iint_D \frac{\partial P}{\partial y}(x,y)\,dx\,dy$$

Nell'altro caso (**) uso una funzione ausiliaria

$$F(x,y) \doteq \int_{\gamma_{x,y}} P\,dx = \int_a^y P(\alpha(t),t)\,\alpha'(t)\,dt + \int_{\alpha(y)}^x P(s,y)\,ds$$



$$\partial_x F(x,y) = P(x,y)$$

$$\partial_y F(x,y) = \cancel{\boxed{P(\alpha(y),y)\,\alpha'(y)}}\ \cancel{\boxed{-P(\alpha(y),y)\,\alpha'(y)}} + \int_{\alpha(y)}^x \partial_y P(s,y)\,ds$$

$$= \int_{\alpha(y)}^x \partial_y P(s,y)\,ds$$

$$\int_{\alpha(y+\varepsilon)}^{x} P(s,y+\varepsilon)\,ds = \int_{\alpha(y)+\varepsilon\alpha'(y)+o(\varepsilon)}^{x} P(s,y)+\partial_y P(s,y)\cdot\varepsilon + o(\varepsilon)\,ds$$

$$= \underbrace{\int_{\alpha(y)}^{x} P(s,y)\,ds}_{\uparrow} \underbrace{- \varepsilon P(\alpha(y),y)\,\alpha'(y) + o(\varepsilon)}_{\int_{\alpha(y)+\varepsilon\alpha'(y)+o(\varepsilon)}^{\alpha(y)} P(s,y)\,dy} + \varepsilon \int_{\alpha(y)}^{x} \partial_y P(s,y)\,ds + o(\varepsilon)$$

Vedi ✪

$$\partial_x F = P(x,y)$$

$$\partial_y F = -\int_{\alpha(y)}^{x} \partial_y P(s,y)\,ds$$

sono continue

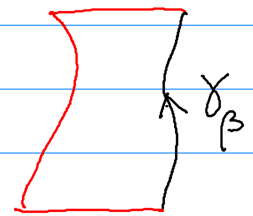
$\partial_x F\,dx + \partial_y F\,dy$ è esatta

$$\int_{+\partial D} \partial_x F\,dx + \partial_y F\,dy = 0$$

$$\int_{+\partial D} P\,dx + \iint \partial_y P\,dx\,dy = 0$$

(A) $\displaystyle\int_{+\partial D} \partial_x F\,dx = \int_{+\partial D} P\,dx$

(B) $\displaystyle\int_{+\partial D} \partial_y F\,dy = \iint_D \partial_y P\,dx\,dy$



fine dim la prossima volta

✪Nota: $G(z,w,x) \doteq \int_z^x P(s,w)\,ds$

$$\partial_y G(\alpha(y),y,x) = \partial_1 G(\alpha(y),y,x)\,\alpha'(y) + \partial_2 G(\alpha(y),y,x) = -P(\alpha(y),y)\,\alpha'(y) + \int_{\alpha(y)}^{x} \partial_y P(s,y)\,ds$$