26 marzo 2021

$$\iint_S \vec{F}\cdot\vec{n}\, d\sigma = \iint_D \vec{F}\circ\phi \frac{\phi_x \wedge \phi_y}{\cancel{|\phi_x\wedge\phi_y|}} \cancel{|\phi_x\wedge\phi_y|}\, dx\,dy$$

$\vec{n}$ $\quad d\sigma$

$S = \{ z = 2 - x^2 - y^2 ,\quad z \geq 0 \}$

$\vec{F} = (x^2, y^2, z)$, $\quad \vec{n}\cdot\vec{e}_3 > 0$

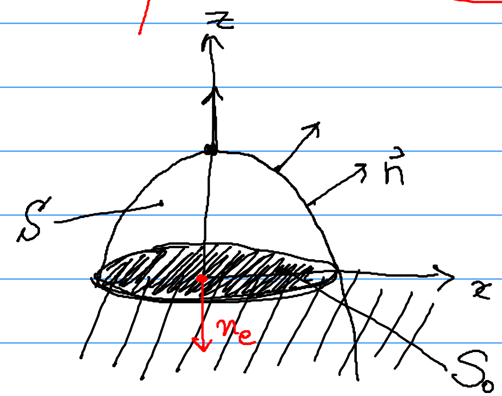
calcolo diretto

$z = 2 - x^2 - y^2 \geq 0$

$D = \{ x^2 + y^2 \leq 2 \}$

$\phi : D \xrightarrow{\sim} S$

$\begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} x \\ y \\ 2-x^2-y^2 \end{pmatrix}$



$\phi_x = (1, 0, -2x)$

$\phi_y = (0, 1, -2y)$

$$\phi_x \wedge \phi_y = \begin{vmatrix} e_1 & e_2 & e_3 \\ 1 & 0 & -2x \\ 0 & 1 & -2y \end{vmatrix} = +2x\, e_1 + 2y\, e_2 + e_3$$

$$\iint_S \vec{F}\cdot\vec{n}\, d\sigma = \iint_D \cancel{2x^3} + \cancel{2y^3} + (2-x^2-y^2)\, dx\,dy = \iint_D 2 - x^2 - y^2\, dx\,dy =$$

$$= \int_0^{2\pi}\int_0^{\sqrt{2}} (2 - r^2)\, r\, dr\, d\theta = 2\pi \left(\left[r^2\right]_0^{\sqrt{2}} - \left[\frac{r^4}{4}\right]_0^{\sqrt{2}} \right) = 2\pi\,(2-1)$$

$V = \{ (x,y,z) :\quad 0 \leq z \leq 2 - x^2 - y^2 \}$ $\qquad \partial V = S \cup S_0$

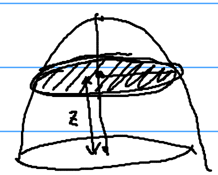
$$\underbrace{\iiint_V \operatorname{div} F\, dx\,dy\,dz}_{} = \iint_{\partial V} \vec{F}\cdot\vec{n}_e\, d\sigma = \boxed{\iint_S \vec{F}\cdot\vec{n}_e\, d\sigma} + \cancel{\iint_{S_0} \vec{F}\cdot\vec{n}_e\, d\sigma}$$

→ è nullo

$\operatorname{div} F = 2x + 2y + 1$

dispari

$$\iiint_V \cancel{2x}\,\cancel{2y} + 1\, dx\,dy\,dz = \iiint_V 1\, dx\,dy\,dz = \int_0^2 \underset{\pi R_z^2}{\text{Area}(V_z)}\, dz = \pi \int_0^2 \underset{R_z^2}{(2-z)}\, dz =$$



$$= \pi \left[-\frac{(2-z)^2}{2} \right]_0^2 = 2\pi$$

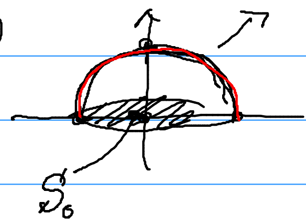
$$\iint_S \vec{F}\cdot\vec{n}\, d\sigma \qquad (\vec{n}\cdot\vec{e}_3 > 0)$$

$S = \{x^2+y^2+z^4 = 1 \ , \ z \geq 0\}$

$F = (1,1,1)$

$V = \{ z \geq 0 \ , \ x^2+y^2+z^4 \leq 1 \}$

su S_0, $n_e = (0,0,-1)$



$\partial V = S \cup S_0$

$$0 = \cancel{\iiint_V \operatorname{div} F} = \boxed{\iint_S F\cdot n_e\, d\sigma} + \iint_{S_0} \vec{F}\cdot n_e\, d\sigma$$

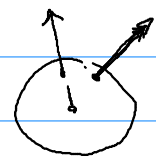
$D = \{x^2+y^2 \leq 1\}$

$$\iint_{S_0} \vec{F}\cdot n_e = \iint_D -1\, dx\,dy = -\pi$$

$$\iint_S \vec{F}\cdot n_e = \pi$$

$$F(x,y,z) = \left(\frac{x}{r^3}, \frac{y}{r^3}, \frac{z}{r^3} \right) \qquad r = \sqrt{x^2+y^2+z^2}$$

F è un campo radiale, a simm. sferica, $\operatorname{div} \vec{F} = 0$



$$\partial_x F = \frac{1}{r^3} + x\, \partial_x\left(\frac{1}{r^3}\right) \qquad \partial_x r = \frac{x}{r}$$

$$\partial_x\left(\frac{1}{r^3}\right) = -\frac{3r^2\,\partial_x r}{r^6} = -\frac{3x}{r^5}$$

$$= \frac{1}{r^3} - \frac{3x^2}{r^5}$$

$$\partial_y F = \frac{1}{r^3} - \frac{3y^2}{r^5}$$

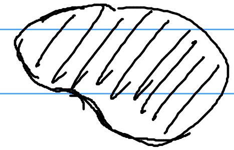
$$\partial_z F = \frac{1}{r^3} - \frac{3z^2}{r^5}$$

$$\operatorname{div} F = \partial_x F + \partial_y F + \partial_z F = \frac{3}{r^3} - \frac{\overbrace{x^2+y^2+z^2}^{r^2}}{r^5} \equiv 0$$

$$F(x,y,z)=\left(\frac{x}{r^3},\frac{y}{r^3},\frac{z}{r^3}\right) \qquad \vec{F}: \mathbb{R}^3\setminus\{\underline{0}\} \longrightarrow \mathbb{R}^3$$

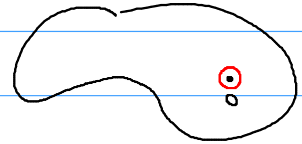
$$\iint_{\partial V} \vec{F}\cdot\vec{n}_e\, d\sigma = \begin{cases} 0 & \text{se } \underline{0}\notin \overline{V} \quad (1) \\ 4\pi & \text{se } \underline{0}\in V \quad (2) \end{cases}$$

V aperto con bordo regolare (a tratti)



(1) $\displaystyle\iint_{\partial V} \vec{F}\cdot\vec{n}_e\, d\sigma = \iiint_V \operatorname{div}\vec{F}\, dxdydz = 0$

$U=\overline{V}\setminus B_\delta(0)$



(2) $\displaystyle\iint_{\partial U} \vec{F}\cdot\vec{n}_e\, d\sigma = \iiint_U \operatorname{div}\vec{F} = 0$

$$\iint_{\partial V}\vec{F}\cdot\vec{n}_e - \iint_{\partial B_\delta(0)} \vec{F}\cdot\vec{n}_e$$

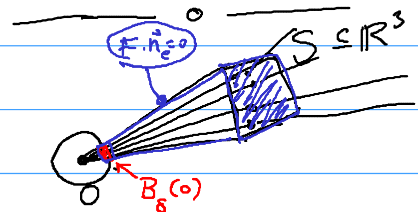
$$\iint_{\partial B_\delta(0)} \frac{1}{\delta^2}\, d\sigma = \frac{1}{\delta^2}\,4\pi\delta^2 = 4\pi$$

su $\partial B_\delta(0)$: $\displaystyle F\cdot n_e = \left(\frac{x}{r^3},\frac{y}{r^3},\frac{z}{r^3}\right)\cdot\left(\frac{x}{r},\frac{y}{r},\frac{z}{r}\right) = \frac{x^2+y^2+z^2}{r^4} = \frac{1}{r^2} = \frac{1}{\delta^2}$

da cui

$$\iint_{\partial V}\vec{F}\cdot\vec{n}_e = 4\pi$$

Angolo solido



$S\subseteq\mathbb{R}^3$ superficie regolare

$V=$ "cono generato da S" unila loro angolo solido generato da S

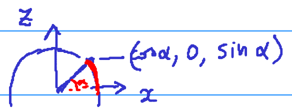
$\Sigma_\delta=\partial B_\delta(0)\cap V$

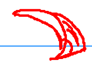
$$\iint_{\Sigma_\delta}\vec{F}\cdot\vec{n} = \iint_S \vec{F}\cdot\vec{n}\, d\sigma$$

misura dell'angolo solido V

Esercizio: Calcolare $\displaystyle\iint_{S_\alpha}\vec{F}\cdot\vec{n}\, d\sigma$ dove F è come sopra

$$S_\alpha=\left\{x^2+y^2+z^2=a^2,\ \begin{matrix} z\geq 0 \\ z\cos\alpha - x\sin\alpha<0\end{matrix}\right\}$$





$\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}$

<u>Esercizio</u>: $\vec{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ $\vec{F}(x) = \varphi(r)\vec{x}$

$r = \sqrt{\sum_{i=1}^{n} x_i^2}$

Quale condizione deve soddisfare φ in modo che $\operatorname{div} F = 0$

— o —

Rotore e NABLA - CALCOLO

$$\operatorname{rot} \vec{F} = \begin{vmatrix} e_1 & e_2 & e_3 \\ \partial_1 & \partial_2 & \partial_3 \\ F_1 & F_2 & F_3 \end{vmatrix} = e_1(\partial_2 F_3 - \partial_3 F_2) + e_2(\partial_3 F_1 - \partial_1 F_3) + e_3(\partial_1 F_2 - \partial_2 F_1)$$

$\boxed{\text{se } \overset{F}{\varphi} \text{ è } C^2}$

- • $\operatorname{rot}(\nabla\varphi) = 0$
- •• $\operatorname{div}(\operatorname{rot}\vec{F}) = 0$

$\nabla \times (\nabla\varphi) = 0$	$\nabla\varphi = \operatorname{grad}\varphi$
	$\vec{\nabla} \cdot \vec{F} = \operatorname{div} F$
$\nabla \cdot (\nabla \times F) = 0$	$\vec{\nabla} \times \vec{F} = \operatorname{rot} F$

notazione dei fisici

• è $F_i = \partial_i \varphi$

$$\operatorname{rot}(\nabla\varphi) = e_1(\partial_2\partial_3\varphi - \partial_3\partial_2\varphi) + e_2(\partial_3\partial_1\varphi - \partial_1\partial_3\varphi) + e_3(\partial_1\partial_2\varphi - \partial_2\partial_1\varphi)$$

è zero per Schwartz

••

$$\operatorname{div}(\operatorname{rot} F) = \partial_1(\partial_2 F_3 - \partial_3 F_2) + \partial_2(\partial_3 F_1 - \partial_1 F_3) + \partial_3(\partial_1 F_2 - \partial_2 F_1)$$

$$= 0$$

— o —

<u>Oss</u>: Se $\vec{F} : \Omega \longrightarrow \mathbb{R}^3$ campo C^1 t.c. $\operatorname{rot}\vec{F} = 0 \Longrightarrow F = \nabla\varphi$

Ω sempl. connesso $\Updownarrow$

$\Omega \subseteq \mathbb{R}^3$ ($F_1 dx + F_2 dy + F_3 dz$ è chiusa)

<u>Domanda</u>: Se $G : \Omega \longrightarrow \mathbb{R}^3$ campo C^1 t.c. $\operatorname{div} G = 0 \;\overset{?}{\Longrightarrow}\; \exists \vec{F} : \; G = \operatorname{rot}\vec{F}$

__Risposta__: In generale no, ma se $\Omega = I_1 \times I_2 \times I_3$ allora sì

— o —

Controesempio: $G = \left(\frac{x}{r^3}, \frac{y}{r^3}, \frac{z}{r^3} \right)$ $\quad \operatorname{div} G = 0$ $\quad$ ma G non è un rotore

__Dim__ Procedo per assurdo: suppong $F : \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}^3$

$\operatorname{rot} F = G$

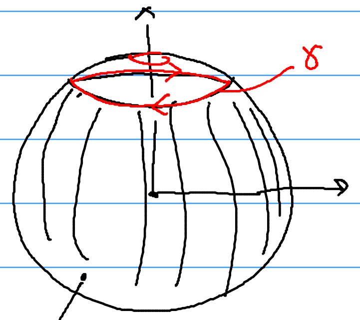
Formula di Stokes

$$\iint_{S_a} \vec{G} \cdot \vec{n}_e \, d\sigma = \iint_{S_a} \operatorname{rot} \vec{F} \cdot \vec{n} \, d\sigma = \oint_\gamma \vec{F} \cdot \vec{T} \, ds \qquad \boxed{a \to 1}$$

$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$

$$\underbrace{\iint_{S_1} \vec{G} \cdot \vec{n}_e \, d\sigma}_{= 4\pi} = 0$$

assurdo.



$S_a = \{ x^2 + y^2 + z^2 = 1, \ z \leq a \}$

$a \in (0,1)$

— o —

Se $G : \Omega \to \mathbb{R}^3$ campo C^1, $\operatorname{div} G = 0$, $\Omega = I_1 \times I_2 \times I_3$ $\Big\} \Longrightarrow \exists F \in C^2 : G = \operatorname{rot} F$

$\boxed{F \text{ si chiama potenziale vettore}}$

__Oss__: le F $\tilde{F}$ pot. vettori per G $\iff \operatorname{rot}(F - \tilde{F}) = 0 \iff F = \tilde{F} + \nabla\varphi$ ho molta libertà di scelta

— o —

$G = \operatorname{Rot} F$

$$\begin{cases} G_1 = \partial_2 F_3 - \partial_3 F_2 & (1) \\ G_2 = \boxed{\cancel{\partial_3 F_1}} - \partial_1 F_3 & (2) \\ G_3 = \partial_1 F_2 - \boxed{\cancel{\partial_2 F_1}} & (3) \end{cases}$$

$$\boxed{F_1 = 0 \qquad \text{(ansatz)}}$$

$$G_2 = -\partial_1 F_3 \qquad (2)$$
$$G_3 = \partial_1 F_2 \qquad (3)$$

$$F_3(x,y,z) - \overbrace{F_3(x_0,y,z)}^{f(y,z)} = \int_{x_0}^{x} \partial_1 F_3(t,y,z)\,dt = -\int_{x_0}^{x} G_2(t,y,z)\,dt$$

$$F_3(x,y,z) = -\int_{x_0}^{x} G_2(t,y,z)\,dt + \cancel{f(y,z)}$$

$\boxed{\text{scelgo } f=0}$

$$F_2(x,y,z) = \int_{x_0}^{x} G_3(t,y,z)\,dt + g(y,z)$$

le metto nella Cond (1)

$$\boxed{G_1 = \partial_2 F_3 - \partial_3 F_2 \qquad (1)}$$

$$\cancel{G_1} = \partial_2 F_3 - \partial_3 F_1 = -\partial_2 \int_{x_0}^{x} G_2(t,y,z)\,dt - \partial_3 \int_{x_0}^{x} G_3(t,y,z)\,dt - \partial_3 g$$

$$= \int_{x_0}^{x} \left[-\partial_2 G_2(t,y,z) - \partial_3 G_3(t,y,z)\right] dt - \partial_3 g$$

$\operatorname{div} G = 0$

$\partial_1 G_1 + \partial_2 G_2 + \partial_3 G_3 = 0$

$$= \int_{x_0}^{x} \partial_1 G_1(t,y,z)\,dt - \partial_3 g = \cancel{G_1(x,y,z)} - G_1(x_0,y,z) - \partial_3 g$$

Basta prendere $-\partial_3 g = G_1(x_0,y,z)$ cioè $g(y,z) \doteq -\int_{z_0}^{z} G_1(x_0,y,s)\,ds$

Riassumendo il campo

$$\begin{cases} F_1 = 0 \\ F_2 = \int_{x_0}^{x} G_3(t,x,y)\,dt - \int_{z_0}^{z} G_1(x_0,y,s)\,ds \\ F_3 = -\int_{x_0}^{x} G_2(t,y,z)\,dt \end{cases}$$

soddisfa $\boxed{\operatorname{rot} \vec{F} = \vec{G}}$

dove $x_0 \in I_1$, $z_0 \in I_3$ arbitrari

Esercizio: trovare un potenziale vettore per il campo $F = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ (campo costante) se $F = (F_1, F_2, F_3)$ è tale che $\operatorname{rot} F = G$

$\varphi(x,y,z) := \int_{x_0}^{x} F_1(t,y,z)\,dt$ $\quad \varphi$ è ben definita

$\tilde{F} \doteq F - \nabla\varphi$ ha come prima componente $F_1 - \partial_x \varphi = F_1 - F_1 = 0$