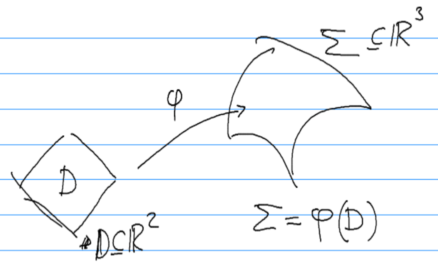
19 mar 2021



$\Sigma=\varphi(D)$

$H^2(\Sigma)$

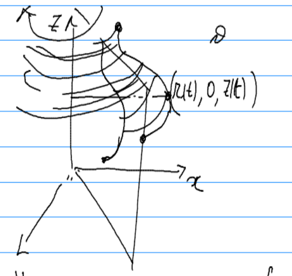
"Area$(\Sigma)=\iint_D J_\varphi(u,v)\,du\,dv$

$J_\varphi(u,v)=|\partial_u\varphi\wedge\partial_v\varphi|$

$=\sqrt{\det(D\varphi^T\cdot D\varphi)}$ (*)

Esercizio: scrivere J_φ nel caso di una superficie di rotazione (risp. a z)

$$\varphi(t,\theta)=\begin{pmatrix} r(t)\cos\theta \\ r(t)\sin\theta \\ z(t)\end{pmatrix}$$



$\partial_t\varphi=?$

$\partial_\theta\varphi=?$

$\underbrace{|\partial_t\varphi\wedge\partial_\theta\varphi|}=?$

e usare la formula ottenuta per calcolare la superficie di una sfera / un toro

$$\partial_t\varphi=\begin{pmatrix} r'(t)\cos\theta \\ r'(t)\sin\theta \\ z'(t)\end{pmatrix}\qquad \partial_\theta\varphi=\begin{pmatrix} -r(t)\sin\theta \\ r(t)\cos\theta \\ 0\end{pmatrix}$$

$$\partial_t\varphi\wedge\partial_\theta\varphi=\begin{vmatrix} e_1 & e_2 & e_3 \\ r'c & r's & z' \\ -rs & rc & 0\end{vmatrix}=$$

$$=e_1(-rcz')-e_2(rsz')+e_3\underbrace{(rr'c^2+rr's^2)}_{rr'}$$

$$=e_1(-r(t)\cos\theta\, z'(t))-e_2(r(t)\sin\theta\, z'(t))+e_3\, r(t)r'(t)$$

$$|\partial_t\varphi\wedge\partial_\theta\varphi|=\sqrt{r^2r'^2+r^2z'^2}=\underbrace{r\sqrt{r'^2+z'^2}}$$

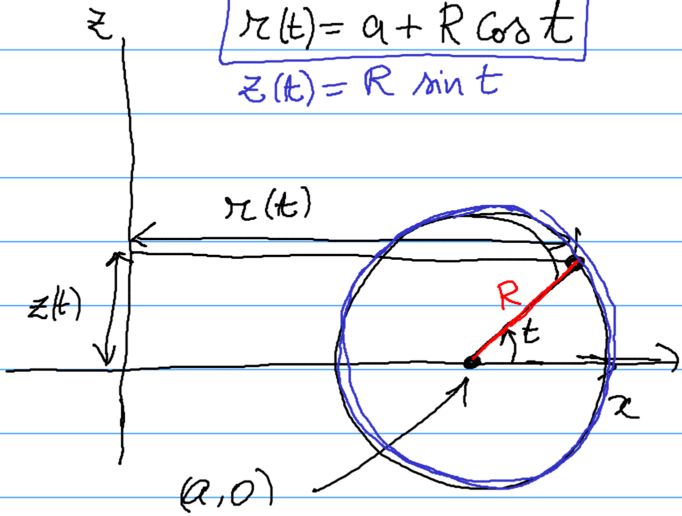
Esempio$_0$: (verifica x esercizio) Calcolo superficie della sfera $4\pi r^2$

Esempio$_1$: Calcolo superficie di un toro

$$T\Leftrightarrow\begin{cases} x(t,\theta)=(a+R\cos t)\cos\theta \\ y(t,\theta)=(a+R\cos t)\sin\theta \\ z(t,\theta)=R\sin t\end{cases}$$

$\boxed{r(t)=a+R\cos t}$

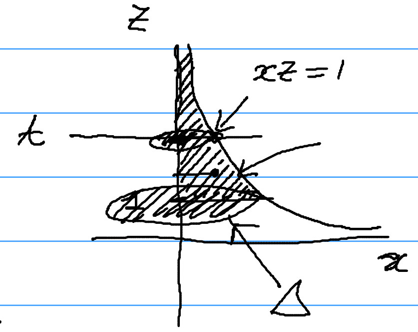
$z(t)=R\sin t$



$(t,\theta)\in[0,2\pi]\times[0,2\pi]$

$$H^2(T)=\int_0^{2\pi}\int_0^{2\pi}(a+R\cos t)\sqrt{(R\sin t)^2+R^2\cos^2 t}\;d\theta\,dt$$

$$= \int_0^{2\pi}\int_0^{2\pi} (a + R\cos t) R \, d\theta \, dt = 4\pi^2 a R$$



$$D = \left\{ (x,y,z) : \quad \sqrt{x^2+y^2} \leq \frac{1}{z} \,, \quad z \geq 1 \right\}$$

$$\mathrm{Vol}(D) = \int_1^{+\infty} \mathrm{Area}(D_t) \, dt \qquad D_t = D \cap \{z = t\}$$

$$\mathrm{Area}(D_t) = \pi \frac{1}{t^2}$$

$$= \pi \int_1^{+\infty} \frac{1}{t^2} \, dt = \pi$$

$$\partial D = \Delta \cup \Sigma \qquad \Delta = \left\{ (x,y,z) : \begin{array}{l} z = 1 \\ \sqrt{x^2+y^2} \leq 1 \end{array} \right\}$$

$$\Sigma = \left\{ (x,y,z) : \begin{array}{l} z \geq 1 \\ \sqrt{x^2+y^2} = \frac{1}{z} \end{array} \right\}$$

$$H^2(\partial D) = H^2(\Delta) + H^2(\Sigma)$$

$$= \pi + ?$$

Uso la formula vista sopra

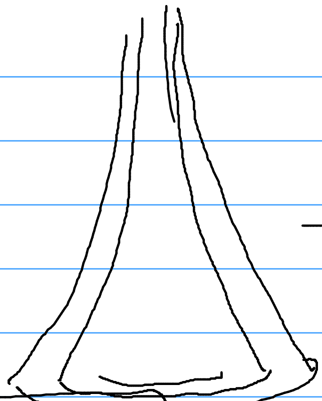
$$z(t) = t \qquad z' = 1$$

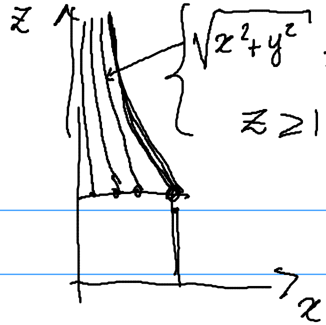
$$r(t) = \frac{1}{t} \qquad r' = -\frac{1}{t^2}$$

$$H^2(\Sigma) = \int_1^{+\infty} dt \int_0^{2\pi} d\theta \left[\frac{1}{t} \sqrt{1^2 + \frac{1}{t^4}} \right] \geq \int_0^{2\pi} d\theta \left(\underbrace{\int_1^{+\infty} \frac{1}{t} \, dt}_{\to +\infty} \right)$$

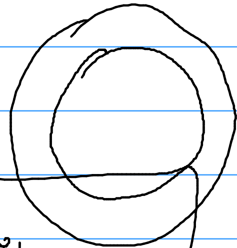
$$\frac{1}{t} \cdot \sqrt{1 + \frac{1}{t^4}} \geq \frac{1}{t}$$

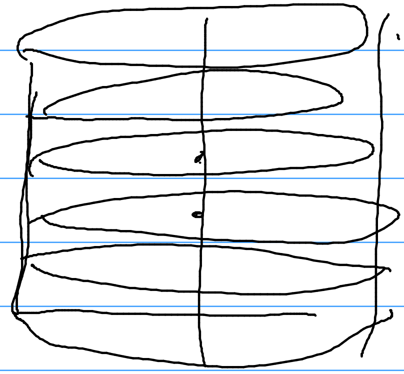
$$\mathrm{Vol}(D) < +\infty \quad \text{ma} \quad H^2(\partial D) = +\infty$$





$$\left\{\begin{array}{l}\sqrt{x^2+y^2} = \dfrac{t}{z}\\ z\geq 1\end{array}\right\} = \tilde{D}_t \qquad t\in[0,1]$$





$$\boxed{Vol(D) = \int H^2(\tilde{D}_t)\,dt \quad \text{NO}}$$

FORMULA DI COAREA

Se $D\subseteq\Omega$ aperto, $\Phi\in C^1(\Omega)$, $\nabla\phi\neq 0$ su D
$(\Omega\subseteq\mathbb{R}^n)$

$$\Sigma_t = D\cap\{\phi(x)=t\}$$

$$Vol(D) = \int_{\mathbb{R}} dt \int_{\Sigma_t} \frac{1}{\|\nabla\phi\|}\, dH^{n-1}$$

misura $(n-1)$-dimensionale su Σ_t

Analogamente

$$\int_D f\,d\bar{x} = \int_{\mathbb{R}} dt \int_{\Sigma_t} \frac{f}{\|\nabla\phi\|}\, dH^{n-1}$$

Oss: In molti casi famosi $\|\nabla\phi\|$ è costante su Σ_t

per esempio $\phi(x,y,z) = z$

$\phi(x,y,z) = \sqrt{x^2+y^2+z^2}$

TEOREMA DELLA DIVERGENZA

$D\subseteq\mathbb{R}^n$ dominio con bordo regolare (regolare a pezzi); F campo vettor C^1

$$\int_{\partial D} \vec{F}\cdot\vec{n} = \int_D \operatorname{div} F$$

$F = (F^1,\dots,F^n)$

$\operatorname{div} F = \sum_{i=1}^{n} \partial_i F^i$

Sia H funzione $\boxed{\text{omogenea di grado } \alpha}$ $H: \mathbb{R}^3 \to \mathbb{R}$; $\boxed{\Delta H = 1}$

calcolare

$$\alpha \iint_{\partial B_1(0)} H = \iint_{\partial B_1(0)} \nabla H(x) \cdot x = \iint_{\partial B_1(0)} \nabla H \cdot n$$

integrale di superficie di una funz. scalare

$$= \iiint_{B_1(0)} \operatorname{div}(\nabla H)\, dx\,dy\,dz = \frac{4\pi}{3}$$

H α-omogenea $\Rightarrow$ $\nabla H(x) \cdot x = \alpha H(x)$

$\nabla H(x) \cdot x = \frac{\partial H}{\partial n}$ $\quad x \in B_1(0)$, $|x| = 1$

$$\iint_{\partial B_1(0)} H = \frac{4\pi}{3\alpha}$$

$\operatorname{div}(\nabla H) = \Delta H$

FUNZIONI ARMONICHE (in dim $=2$) $\quad \Omega \subseteq \mathbb{R}^2$ aperto

$u \in C^2(\Omega)$. Sono equivalenti

(i) $\Delta u \equiv 0$ su Ω

(ii) $\forall\, P_0 \in \Omega$, $r > 0$ t.c. $B_r(P_0) \subseteq \Omega$ si ha $\overbrace{\frac{1}{2\pi r}\int_{\partial B_r(P_0)} u}^{\varphi(r)} = u(P_0)$

Dim [(i) $\Rightarrow$ (ii)] S.P.G. $P_0 = 0$ $\quad P_0 = (x_0, y_0)$

$$\varphi(r) := \frac{1}{2\pi r}\int_{\partial B_r(0)} u = \frac{1}{2\pi \cancel{r}}\int_0^{2\pi} u(r\cos\theta, r\sin\theta)\cdot \cancel{r}\, d\theta$$

$\overset{+}{\partial B_r} \Leftrightarrow \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix}$

$\boxed{\operatorname{div} \nabla u = \Delta u \overset{(i)}{=} 0}$

$$\int_\gamma u = \int_a^b u(\gamma(t))\, \underbrace{|\gamma'(t)|}\, dt$$

$$= \frac{1}{2\pi}\int_0^{2\pi} u(r\cos\theta, r\sin\theta)\, d\theta$$

normale esterna $\boxed{n(\theta)}$ $\quad \vec{F} = \nabla u$

$$\varphi'(r) = \frac{1}{2\pi}\int_0^{2\pi} \nabla u(r\cos\theta, r\sin\theta) \cdot \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} d\theta = \frac{1}{2\pi}\int_{\partial B_r} \nabla u \cdot n = \frac{1}{2\pi}\iint_{B_r} \operatorname{div} \nabla u\, dx\,dy = 0$$

$\Rightarrow \varphi$ è costante. $\lim\limits_{r\to 0} \varphi(r) = u(0,0)$ (per continuità) $\Rightarrow$.

$\varphi(r) = \varphi(0,0)$ $\quad \forall\, r > 0$ ammissibile

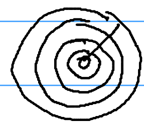
$\boxed{(i) \Leftarrow (ii)}$ Devo verificare $\Delta u = 0$ in ogni punto P_0 $\quad u : \Omega \subseteq \mathbb{R}^2 \to \mathbb{R}$

S.P.G. suppongo $P_0 = 0$, e $u(P_0) = 0$

Osservo che per (ii) $\quad \cancel{\frac{1}{2\pi r}} \int_{\partial B_r(0)} u = 0 \quad \forall\, r > 0$ ammissibile

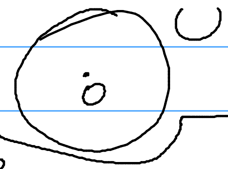
$$\Downarrow$$

$$\iint_{B_r(0)} u(x,y)\,dx\,dy = 0 \xrightarrow{\text{LEMMA}} \text{tesi } \Delta u(0) = 0$$



Lemma: Se u è $C^2(U)$, $u(0,0) = 0$ allora

$$\lim_{r\to 0} \frac{1}{r^4} \iint_{B_r(0)} u(x,y)\,dx\,dy = c\,\Delta u(0) \qquad \boxed{c = \frac{\pi}{8}}$$



Es: fare il calcolo nel caso generale $U \subseteq \mathbb{R}^n$ intorno di 0, $u(0) = 0$

$$\lim_{r\to 0} \frac{1}{r^{n+2}} \int_{B_r(0)} u(\vec{x})\,d\vec{x} = c_n\,\Delta u(0)$$

Dim Lemma: Sviluppo u con Taylor vicino a 0

$$u(x,y) = \underbrace{u(0,0)}_{0} + u_x(0,0)\,x + u_y(0,0)\,y + \boxed{u_{xx}(0,0)\frac{x^2}{2}} + u_{xy}(0,0)\,xy + \boxed{u_{yy}(0,0)\frac{y^2}{2}} + o(x^2+y^2)$$

$$\iint_{B_r(0)} x\,dx\,dy = 0 \qquad \iint_{B_r(0)} y\,dx\,dy = 0 \qquad \iint_{B_r(0)} xy\,dx\,dy = 0 \qquad \text{Per simmetria}$$

$\int_{B_r(0)} |x|^2 \, dx \sim c r^{n+2} \qquad r \to 0$

Per simmetria

$$\iint_{B_r(0)} x^2 \, dx\,dy = \iint_{B_r(0)} y^2 \, dx\,dy = \frac{1}{2} \iint_{B_r(0)} (x^2+y^2) \, dx\,dy = \frac{1}{2} \int_0^{2\pi} \int_0^r \rho^2 \, \rho \, d\rho \, d\theta = \pi \left[\frac{\rho^4}{4} \right]_0^r$$

$$= \frac{\pi}{4} r^4$$

$$\boxed{\iint_{B_r(0)} o(r^2) \, dx\,dy = o(r^4)}$$

$$\omega = o(x^2+y^2) \Longrightarrow \forall \varepsilon > 0 \ \exists R > 0 : \ \omega(x,y) \le \varepsilon (x^2+y^2)$$

$$\frac{1}{r^4} \iint_{B_r(0)} u(x,y) \, dx\,dy = \frac{1}{r^4} \left[u_{xx}(0) \frac{\pi}{8} r^4 + u_{yy}(0) \frac{\pi}{8} r^4 + o(r^4) \right] \xrightarrow[r \to 0]{} (u_{xx} + u_{yy}) \frac{\pi}{8}$$

$\blacksquare$