Lyapunov equations

Lyapunov equation

$$A^*X + XA + Q = 0, \quad Q = Q^* \succeq 0. \tag{L}$$

Special case of the Sylvester equation.
It has a unique solution whenever $\Lambda(A) \cup \Lambda(-A^*) = \emptyset$.
Important case: when $\Lambda(A) \subseteq LHP$ (open left half-plane).

Lemma

Suppose (L) has a unique solution X; then X is Hermitian.

Proof: transpose everything; X^* is another solution.

Lyapunov equation: positivity

Lemma

Suppose A has eigenvalues in the (open) LHP. Then, $Q \succeq 0$ implies $X \succeq 0$, and $Q \succ 0$ implies $X \succ 0$.

Proof Check that

$$X = \int_0^\infty e^{A^* t} Q e^{At} \, \mathrm{d}t.$$

$(\frac{\mathrm{d}}{\mathrm{d}t} e^{A^* t} Q e^{At} = A^* e^{A^* t} Q e^{At} + e^{A^* t} Q e^{At} A$, then integrate both sides).

Lemma

Suppose $Q \succ 0$ and $X \succ 0$. Then, A has eigenvalues in the (open) LHP.

Proof Let $Av = \lambda v$; then $0 < v^* Q v = \dots$

Relation to linear dynamical systems

Alternative statement: to prove that A has all its eigenvalues in the LHP, it is sufficient to exhibit $X \succ 0$ such that $A^*X + XA \prec 0$.

Alternative way to see it: when is the dynamical system $\dot{x}(t) = Ax(t)$ asymptotically stable, i.e., $x(t) \to 0$ for every x_0? Differential equation theory: when $\Lambda(A) \subset LHP$.

Lyapunov's way to prove it: it is sufficient to exhibit $X \succ 0$ such that $A^*X + XA \prec 0$. Once you find it, you can define the energy function $V(t) = x^*Xx$; then $\frac{d}{dt}V(x(t)) \leq 0$, so $x(t)$ cannot escape the (finite) region $\{x \colon V(x) \leq V(x_0)\}$. Strengthening slightly the argument, one gets $x(t) \to 0$ if $X \succ 0$.

Lyapunov (1857–1918) did not have computers, so at that time exhibiting a solution to that equation was typically easier than computing all the eigenvalues of a nonsymmetric matrix A.

Discrete-time version

Many of these results also come in a 'discrete-time' variant; in this case:

Stein's equation

A has all its eigenvalues in the (open) unit disc iff

$$X - A^* X A = Q, \quad Q \succ 0$$

has a solution $X \succ 0$.

Related to the stability of the discrete-time system $x_{t+1} = Ax_t$.

Can be solved with a Bartels-Stewart-like method.
(Skipping some details: a certain quantity has to be cached during the computation so that the cost is $O(n^3)$)
(Actually B-S works for all equations of the kind $AXB + CXD = E$, using QZ factorizations of (A, C) and (D^T, B^T).)

Discrete-time version

Discrete-time version of the integral formula:

If A has all its eigenvalues inside the (open) unit disc then

$$X = \sum_{k=0}^{\infty} (A^*)^k Q A^k.$$

Proof $(I - A^T \otimes A^*) \operatorname{vec}(X) = \operatorname{vec}(Q)$, then use the Neumann series for the inverse.