Newton's method for CARE

$$F(X) = A^T X + XA + Q - XGX$$

$$L_{F,X}(H) = A^T H + HA - HGX - XGH = H(A - GX) + (A - GX)^T H.$$

$$\widehat{L}_{F,X} = (A - GX)^T \otimes I + I \otimes (A - GX)^*.$$

If X_* is the stabilizing solution then $\Lambda(A - GX_*) \subset LHP \implies L_{F,X_*}$ is nonsingular.

Newton's method

For $k = 0, 1, 2, \ldots$

1. Solve $H(A - GX_k) + (A - GX_k)^T H = F(X_k)$ for H;
2. Set $X_{k+1} = X_k - H$.

Newton's method

Note that $H(A - GX_k) + (A - GX_k)^T H = F(X_k)$ is equivalent to

$$X_{k+1}(A - GX_k) + (A - GX_k)^* X_{k+1} = -Q - X_k G X_k.$$

This shows that $A - GX_k$ stable $\implies X_{k+1} \succeq 0$.

Actually, something stronger holds.

Monotonicity of Newton's method

Theorem

Suppose X_0 is chosen such that $\Lambda(A - GX_0) \subset LHP$. Then, $X_1 \succeq X_2 \succeq X_3 \succeq \cdots \succeq X_* \succeq 0$. Moreover, $X_k \to X_*$ quadratically.

Proof (sketch) Coupled induction. Set $A_k := A - GX_k$:

$$(X_k - X_{k+1})A_k + A_k^*(X_k - X_{k+1}) = -(X_k - X_{k-1})G(X_k - X_{k-1})$$
$$(X_* - X_{k+1})A_k + A_k^*(X_* - X_{k+1}) = -(X_* - X_k)G(X_* - X_k)$$

hence A_k stable $\implies X_k \succeq X_{k+1} \succeq X_*$.

$$\begin{aligned}&(X_{k+1} - X_*)A_{k+1} + A_{k+1}^*(X_{k+1} - X_*)\\&\quad = -(X_{k+1} - X_k)G(X_{k+1} - X_k) - (X_{k+1} - X_*)G(X_{k+1} - X_*)\end{aligned}$$

This does not prove immediately that A_{k+1} is stable (because the RHS is not $\prec 0$), but $A_{k+1}v = \lambda v$ with $\operatorname{Re}\lambda \geq 0$ gives $B(X_{k+1} - X_k)v = 0$, hence also $A_k v = \lambda v$.

Newton: wrap-up

Remark Note that the theorem does not include $X_0 \succeq X_1$; anything could happen in the first iteration!

Algorithm

- Use Bass's algorithm to find X_0 such that $A - GX_0$ is stable
- Run Newton iterations until convergence.

Expensive: each iteration requires a Schur form.

Convergence: standard quadratic convergence of Newton's method holds: if the solution is simple (which is the case whenever the Hamiltonian has no imaginary eigenvalues $\iff$ $L_{F,X}$ is invertible), then $\|X_* - X_{k+1}\| \sim \|X_* - X_k\|^2$.

Defect correction / iterative refinement: One final step of Newton can be used to 'correct' an inaccurate algorithm (as long as it's accurate enough to give an initial guess in the correct basin of attraction).