Sign-like methods for CAREs

Matrix sign iteration

$$X_{k+1} = \frac{1}{2}(X_k + X_k^{-1}), \quad X_0 = \mathcal{H}.$$

It is not difficult to see that X_k is Hamiltonian at each step (i.e., $JX_k = -X_k^* J$).

Lemma

- Let M be Hamiltonian. Then M^{-1} is Hamiltonian, too.
- Let M_1, M_2 be Hamiltonian. Then $M_1 + M_2$ is Hamiltonian, too.

(Guiding idea: Hamiltonian matrices are 'like antisymmetric ones': properties that you expect for antisymmetric matrices will often hold for Hamiltonian, too.)

Structure-preserving sign iteration

In machine arithmetic, the X_k won't be exactly Hamiltonian — unless we modify our algorithm to ensure that they are.

Observation: $\mathcal{H}$ is Hamiltonian iff $J\mathcal{H}$ is symmetric.
Rewrite the iteration in terms of $Z_k := JX_k$:

$$Z_{k+1} = \frac{1}{2}(Z_k + JZ_k^{-1}J), \quad Z_0 = J\mathcal{H}.$$

This version preserves symmetry exactly (assuming the method we use for inversion does).

We can incorporate scaling.

Towards doubling

Recall: in the sign iteration, if we set $Y_k = (I - X_k)^{-1}(I + X_k)$, then $Y_{k+1} = -Y_k^2$.

In an ideal world without rounding errors, we could compute $Y_0, Y_1, Y_2, \dots$, and then get the stable invariant subspace as $\ker Y_\infty$ (or, rather, the invariant subspace associated to the n smallest singular values of Y_∞, since in an ideal world without rounding errors it is nonsingular).

We can do something similar, if we work in a suitable format.

Standard Symplectic Form

Goal: write $Y_0 = (I - \mathcal{H})^{-1}(I + \mathcal{H})$ as

$$Y_0 = \begin{bmatrix} I & G_0 \\ 0 & F_0 \end{bmatrix}^{-1} \begin{bmatrix} E_0 & 0 \\ H_0 & I \end{bmatrix}.$$

Trick: this is equivalent to finding M such that

$$M \begin{bmatrix} (I - \mathcal{H}) & (I + \mathcal{H}) \end{bmatrix} = \begin{bmatrix} I & G_0 & E_0 & 0 \\ 0 & F_0 & H_0 & I \end{bmatrix}.$$

And this M must be the inverse of block columns $(1, 4)$.

Structural properties:

- if $\mathcal{H}$ is Hamiltonian, Y_0 is symplectic.
  Proof: via $(I - \mathcal{H})^* J (I - \mathcal{H}) = (I + \mathcal{H})^* J (I + \mathcal{H})$.
- If Y_0 is symplectic, $E_0 = F_0^*, G_0 = G_0^*, H_0 = H_0^*$.
- Moreover, if $G \succeq 0$, $H \succeq 0$, then $G_0 \succeq 0$, $H_0 \preceq 0$ (tedious).

Doubling algorithm

Plan Given $Y_k = \begin{bmatrix} I & G_k \\ 0 & E_k^* \end{bmatrix}^{-1} \begin{bmatrix} E_k & 0 \\ H_k & I \end{bmatrix}$, compute

$$Y_{k+1} = -Y_k^2 = \begin{bmatrix} I & G_{k+1} \\ 0 & E_{k+1}^* \end{bmatrix}^{-1} \begin{bmatrix} E_{k+1} & 0 \\ H_{k+1} & I \end{bmatrix}.$$

Similar to the 'inverse-free sign method' described earlier.

The swap: If $Y_k = \mathcal{M}_k^{-1}\mathcal{N}_k$, then $-Y_k^2 = -\mathcal{M}_k^{-1}\mathcal{N}_k\mathcal{M}_k^{-1}\mathcal{N}_k = \mathcal{M}_k^{-1}\widehat{\mathcal{M}}_k^{-1}\widehat{\mathcal{N}}_k\mathcal{N}_k = (\widehat{\mathcal{M}}_k\mathcal{M}_k)^{-1}(\widehat{\mathcal{N}}_k\mathcal{N}_k)$, where $\widehat{\mathcal{M}}_k, \widehat{\mathcal{N}}_k$ satisfy $\widehat{\mathcal{M}}_k^{-1}\widehat{\mathcal{N}}_k = -\mathcal{N}_k\mathcal{M}_k^{-1}$, i.e.,

$$\begin{bmatrix} \widehat{\mathcal{M}}_k & \widehat{\mathcal{N}}_k \end{bmatrix} \begin{bmatrix} \mathcal{N}_k \\ \mathcal{M}_k \end{bmatrix} = 0.$$

Doubling: the swap

$$\begin{bmatrix} I & \widehat{G}_k & \widehat{E}_k & 0 \\ 0 & \widehat{F}_k & \widehat{H}_k & I \end{bmatrix} \begin{bmatrix} E_k & 0 \\ H_k & I \\ I & G_k \\ 0 & E_k^* \end{bmatrix} = 0$$

holds if

$$\begin{aligned} \begin{bmatrix} \widehat{G}_k & \widehat{E}_k \\ \widehat{F}_k & \widehat{H}_k \end{bmatrix} &= - \begin{bmatrix} E_k & 0 \\ 0 & E_k^* \end{bmatrix} \begin{bmatrix} H_k & I \\ I & G_k \end{bmatrix}^{-1} \\ &= \begin{bmatrix} E_k & 0 \\ 0 & E_k^* \end{bmatrix} \begin{bmatrix} G_k(I - H_k G_k)^{-1} & -(I - G_k H_k)^{-1} \\ -(I - H_k G_k)^{-1} & H_k(I - G_k H_k)^{-1} \end{bmatrix}. \end{aligned}$$

Doubling: the formulas

Putting everything together,

$$
\begin{aligned}
\begin{bmatrix} E_{k+1} & 0 \\ H_{k+1} & I \end{bmatrix} &= \begin{bmatrix} -E_k(I - G_k H_k)^{-1} & 0 \\ E_k^* H_k (I - G_k H_k)^{-1} & I \end{bmatrix} \begin{bmatrix} E_k & 0 \\ H_k & I \end{bmatrix} \\
&= \begin{bmatrix} -E_k(I - G_k H_k)^{-1} E_k & 0 \\ H_k + E_k^* H_k (I - G_k H_k)^{-1} E_k & I \end{bmatrix}
\end{aligned}
$$

and an analogous computation gives E_{k+1}^*, G_{k+1}:

Structured doubling algorithm

$$
\begin{aligned}
E_{k+1} &= -E_k(I - G_k H_k)^{-1} E_k, \\
G_{k+1} &= G_k + E_k G_k (I - H_k G_k)^{-1} E_k^*, \\
H_{k+1} &= H_k + E_k^* H_k (I - G_k H_k)^{-1} E_k.
\end{aligned}
$$

SDA: details

Note that (even when the series does not converge)

$$G_k(I - H_k G_k)^{-1} = G_k + G_k H_k G_k + G_k H_k G_k H_k G_k + \cdots = (I - G_k H_k)^{-1} G_k,$$

and this matrix is symmetric. If $G_k = B_k B_k^*$, then it can also be rewritten as $B_k(I - B_k^* H_k B_k)^{-1} B_k^*$ (inverting a symmetric matrix).

Monotonicity If $H_k \preceq 0$ then $G_k(I - H_k G_k)^{-1} \succeq 0$. Hence, $0 \preceq G_0 \preceq G_1 \preceq \dots$, and $0 \succeq H_0 \succeq H_1 \succeq H_2 \succeq \dots$

Cost As much as a $2n \times 2n$ inversion $M^{-1}N$, if you put everything together. Unlike the sign algorithm, we have a bound $\sigma_{\min}(I - H_k G_k) \geq 1$ (because $G_k \succeq 0$, $H_k \preceq 0$).

SDA: the dual equation

To analyze convergence, we need to introduce another matrix. Let Y be the matrix such that

$$\mathcal{H}\begin{bmatrix}-Y\\ I\end{bmatrix} = \begin{bmatrix}A & -G\\ -Q & -A^*\end{bmatrix}\begin{bmatrix}-Y\\ I\end{bmatrix} = \begin{bmatrix}-Y\\ I\end{bmatrix}\widehat{\mathcal{R}}$$

is the anti-stable invariant subspace of $\mathcal{H}$, i.e., $\Lambda(\widehat{\mathcal{R}}) \subset RHP$.

$\begin{bmatrix}I\\ Y\end{bmatrix}$ spans the stable subspace of $\mathcal{H}^* = -J\mathcal{H}J$; we can prove that the subspace has this form if (A^T, C^T) controllable (typically satisfied).

SDA: convergence (intuitively)

Theorem

In SDA, $E_k \to 0, G_k \to Y, H_k \to -X$. Convergence is quadratic, i.e., $\|H_k + X\| = \mathcal{O}(\rho^{2^k})$ for some $\rho \in [0, 1)$, as $k \to \infty$.

Intuitive view $E_k \to 0$, approximately squared at each time. Hence

$$\mathcal{H}_k = \begin{bmatrix} I & G_k \\ 0 & E_k^* \end{bmatrix}^{-1} \begin{bmatrix} E_k & 0 \\ H_k & I \end{bmatrix}$$

has n eigenvalues $\to 0$ and n that $\to \infty$. $\ker \mathcal{H}_k \approx \begin{bmatrix} I \\ -H_k \end{bmatrix}$, so $-H_k \to X$.

Dually, "$\ker \mathcal{H}_k^{-1}$" (a thing that shouldn't exist. . .) $\approx \begin{bmatrix} -G_k \\ I \end{bmatrix}$, so $G_k \to Y$.

SDA convergence (formally)

Proof some manipulations give

$$\mathcal{H}_0 \begin{bmatrix} I \\ X \end{bmatrix} = (I - \mathcal{H})^{-1}(I + \mathcal{H}) \begin{bmatrix} I \\ X \end{bmatrix} = \begin{bmatrix} I \\ X \end{bmatrix} (I - \mathcal{R})^{-1}(I + \mathcal{R}).$$

where $\mathcal{S} = (I - \mathcal{R})^{-1}(I + \mathcal{R})$ has eigenvalues in the unit circle. Thus

$$\begin{bmatrix} I & G_k \\ 0 & E_k^* \end{bmatrix} \begin{bmatrix} I \\ X \end{bmatrix} = \begin{bmatrix} I \\ X \end{bmatrix} \begin{bmatrix} E_k & 0 \\ H_k & I \end{bmatrix} \begin{bmatrix} I \\ X \end{bmatrix} \mathcal{S}^{2^k}.$$

which implies

$$E_k = (I + G_k X)\mathcal{S}^{2^k},$$
$$H_k + X = E_k^* X \mathcal{S}^{2^k} = (\mathcal{S}^{2^k})^*(I + X G_k)\mathcal{S}^{2^k} \succeq 0.$$

The same computation on the dual equation gives $G_k \preceq Y$, so G_k is bounded and $E_k \to 0, H_k + X \to 0$ (quadratically as $\mathcal{S}^{2^k}$).