Note Title Lyapunov equations 2023-05-16

$A, Q \in \mathbb{C}^{n\times n}$ $Q = Q^* \geq 0$, we look for $W \in \mathbb{C}^{n \times n}$ s.t.

$$A^* W + WA + Q = 0 \qquad (L)$$

(variant of $AX - XB = C$, Sylvester equation)

(L) has a unique solution if and only if $\Lambda(A^*) \cap \Lambda(-A) = \emptyset$

Important case: this holds when $\Lambda(A) \subset LHP$.

Lemma: if (L) has a unique solution W, then W is Hermitian

Proof: Transpose everything:

$$0 = W^* A + A^* W^* + Q$$

So W^* solves the same equation (L), and by uniqueness $W = W^*$

Lemma: If $\Lambda(A) \subset LHP$, then W can be written as

$$W = \int_0^\infty \exp(A^* t) Q \exp(At) \, dt$$

Proof: First note that the integral converges:

If $A = V \Lambda V^{-1}$, then $\exp(At) = V \begin{bmatrix} \exp(\lambda_1 t) & & \\ & \ddots & \\ & & \exp(\lambda_n t) \end{bmatrix} V^{-1}$

converges to 0 since $\Lambda(A) \subset LHP$.

(A similar argument w/ continuity should hold for general A)

We start by computing

$$\frac{d}{dt} \exp(A^* t) Q \exp(At) = A^* \exp(A^* t) Q \exp(At) + \exp(A^* t) Q \exp(At) A$$

Integrate both sides:

$$\int_0^\infty \frac{d}{dt} \exp(A^* t)\, Q \exp(At)\, dt = A^* \int_0^\infty \exp(A^* t)\, Q \exp(At)\, dt + \int_0^\infty \exp(A^* t)\, Q \exp(At)\, dt \cdot A$$

$$\text{RHS} = A^* W + W A$$

$$\text{LHS} = \exp(A^* t)\, Q \exp(At) \Big|_0^\infty = 0 - Q$$

Hence the integral expression satisfies (L), $A^* W + WA + Q = 0$

Corollary:

$$W = \int_0^\infty (\exp(At))^* Q \exp(At)\, dt \succcurlyeq 0.$$

If $Q \succcurlyeq 0$, then W solution of $A^* W + WA + Q = 0$ is $W \succcurlyeq 0$

If $Q \succ 0$ " " $W \succ 0$

Lemma: Suppose $Q \succ 0$ and $W \succ 0$. Then, $\Lambda(A) \subset LHP$

Proof: $Av = v\lambda$, then we can write

$$v^* A^* W v + v^* W A v + v^* Q v = 0$$

$$\bar{\lambda} v^* W v + v^* W v \lambda + v^* Q v = 0$$

$$\bar{\lambda} + \lambda = -\frac{v^* Q v}{v^* W v} < 0$$

Remark: we cannot replace $Q \succ 0$, $W \succ 0$ with $Q \succcurlyeq 0$ $W \succcurlyeq 0$: indeed, already for scalars $\bar{a} \cdot 0 + 0 \cdot a = 0$, and we cannot deduce anything on A.

Consider the continuous-time, linear dynamical system

$$\begin{cases} \dot{x}(t) = A \cdot x(t) \\ x(0) = x_0 \end{cases} \qquad x : [0, \infty) \to \mathbb{C}^n \qquad (*)$$

with solution $x(t) = \exp(At) \cdot x_0$

If $\Lambda(A) \subset LHP$, then $\lim_{t \to \infty} x(t) = 0$

In ~ 1900, it was tricky to compute $\Lambda(A)$ directly, so it was a good technique to show explicitly $Q \succ 0$, $W \succ 0$ s.t. $\boxed{A^* W + WA + Q = 0.}$

Also, the resulting W had an interpretation in terms of energy: if we define the function $V(x) = x^* W x$, then one can see that $\frac{d}{dt} V(x(t)) \leq 0$ if $x(t)$ solves $(*)$: indeed,

$$\frac{d}{dt} V(x(t)) = \frac{d}{dt} x(t)^* W x(t) = \left(\frac{d}{dt} x(t)\right)^* W x(t) + x(t)^* W \left(\frac{d}{dt} x(t)\right)$$

$$= x(t)^* A^* W x(t) + x(t)^* W A x(t) = x(t)^* (-Q) x(t) \leq 0$$

Variant: discrete-time dynamical systems

$$\begin{cases} x_{k+1} = A x_k \qquad k = 0, 1, 2, \dots \\ x_0 \in \mathbb{C}^n \end{cases} \qquad x_k = A^k x_0 \qquad (**)$$

The system is asymptotically stable if $\lim_{k \to \infty} x_k = 0$ for all initial values x_0

$\iff \rho(A) < 1$

The discrete-time analogue of our Lyapunov equation is

The Stein equation

$$W - A^* W A = Q \qquad (S)$$

for $A, Q \in \mathbb{C}^{n\times n}$ $\quad Q = Q^* \succeq 0$, unknown $W \in \mathbb{C}^{n\times n}$

If W solves (S) for a certain $Q \succeq 0$, then $V(x) = x^* W x$ satisfies $V(x_{k+1}) \leq V(x_k)$ for any solution of $(**)$:

$$\underbrace{x_k^* W x_k}_{V(x_k)} - \underbrace{x_k^* A^* W A x_k}_{V(x_{k+1})} = \underbrace{x_k^* Q x_k}_{\geq 0}$$

	Continuous	Discrete
Dyn. system	$\dot{x} = Ax$	$x_{k+1} = Ax_k$
Stability cond.	$\Lambda(A) \subset LHP$	$\Lambda(A) \subset$ unit disc
Equation that gives energy functional	$A^*W + WA + Q = 0$, $Q \succeq 0$	$W - A^*WA = Q$
Expression for sol.	$W = \int_0^\infty \exp(A^*t)\, Q \exp(At)\, dt$	$W = \sum_{k=0}^{\infty} (A^*)^k Q A^k$

Lemma: let $A \in \mathbb{C}^{n\times n}$ s.t. $\Lambda(A) \subset$ unit disc, $Q \succeq 0$, then

$$W = \sum_{k=0}^{\infty} (A^*)^k Q A^k \quad \text{satisfies} \quad W - A^* W A = Q.$$

Proof:

$$W = Q + \sum_{k=1}^{\infty} (A^*)^k Q A^k = Q + A^* \left(\sum_{h=0}^{\infty} (A^*)^h Q A^h \right) A = Q + A^* W A$$

Vectorizing the Stein equation:

$$\mathrm{vec}(W) = \mathrm{vec}(Q) + \mathrm{vec}(A^* W A) = \mathrm{vec}(Q) + \left(A^T \otimes A^*\right)\cdot \mathrm{vec}(W)$$

so

$$(S) \iff \underbrace{\left(I - A^T \otimes A^*\right)}\mathrm{vec}(W) = \mathrm{vec}(Q)$$

Take matrices Q_1, Q_2 s.t. $A^T = Q_1 U_1 Q_1^*$

$A^* = Q_2 U_2 Q_2^*$

Then

$$\left(I - A^T \otimes A^*\right) = \left(Q_1 \otimes Q_2\right)\underbrace{\left(I - U_1 \otimes U_2\right)}_{\text{upper triangular}}\left(Q_1 \otimes Q_2\right)^*$$

upper triangular,

on the diagonal we have

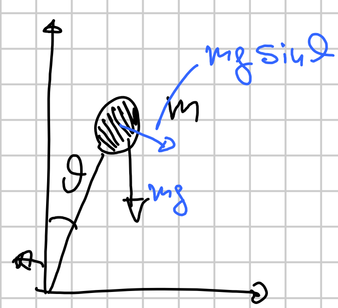
$1 - \lambda_i \bar{\lambda}_j \quad i,j = 1,..n, \quad \lambda_i$ are eigenvalues of A

We have proved the following:

the Stein equation (S) has a unique solution if and only if A does not have two eigenvalues λ_i, λ_j s.t. $\lambda_i \bar{\lambda}_j = 1$, in particular when $\Lambda(A) \subset$ open disc $\{|z| < 1\}$

Control theory:

Example: inverted pendulum



State of the system: $x = \begin{bmatrix} \vartheta \\ \dot{\vartheta} \end{bmatrix}$

Equation of motion: $\cancel{m}\ddot{\vartheta} = \cancel{m}g \sin\vartheta \approx \cancel{m} g \vartheta$

In matrix form, using state $\begin{bmatrix} \vartheta \\ \dot\vartheta \end{bmatrix}$:

$$\dot{x} = \begin{bmatrix} \dot\vartheta \\ \ddot\vartheta \end{bmatrix} = \begin{bmatrix} \dot\vartheta \\ g\vartheta \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ g & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} \vartheta \\ \dot\vartheta \end{bmatrix}}_{x} = Ax$$

$$\boxed{\dot{x} = Ax} \qquad \Lambda(A) = \{\pm\sqrt{g}\} \not\subset LHP$$

$\Rightarrow$ the system is <u>not</u> asymptotically stable

We can add a <u>steering force</u> that acts at the base $u(t)$

$$\ddot\vartheta = g\vartheta + u \qquad u : [0,\infty) \to \mathbb{R}$$

$$\dot{x} = \begin{bmatrix} \dot\vartheta \\ \ddot\vartheta \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ g & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} \vartheta \\ \dot\vartheta \end{bmatrix}}_{x} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} u$$

$$\dot{x} = Ax + Bu \qquad A \in \mathbb{R}^{2\times 2}, \quad B \in \mathbb{R}^{2\times 1}$$

Can I choose a control function $u(t)$ that makes the system asymptotically stable, i.e. $\lim_{t\to\infty} x(t) = 0$ for all x_0

Yes, we can even choose $u(t)$ in a very special form:

$$u(t) = \begin{bmatrix} f_1 & f_2 \end{bmatrix} \cdot \begin{bmatrix} \vartheta(t) \\ \dot\vartheta(t) \end{bmatrix} = Fx(t), \quad \text{with } F \in \mathbb{R}^{1\times 2} \text{ constant.}$$

If I plug $u(t) = Fx(t)$ into the system, I obtain

$$\dot{x}(t) = Ax(t) + Bu(t) = Ax(t) + BFx(t) = (A+BF)\,x(t)$$

The system is asymptotically stable if and only if
$\Lambda(A+BF) \subset LHP$

$$A+BF = \begin{bmatrix} 0 & 1 \\ g & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} [f_1 \;\; f_2] = \begin{bmatrix} 0 & 1 \\ g+f_1 & f_2 \end{bmatrix}$$

$$\det(A+BF-\lambda I) = \lambda(f_2-\lambda) - (g+f_1) = -\lambda^2 + f_2 - (g+f_1)$$

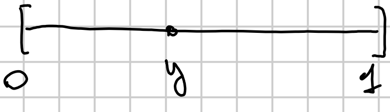
We can choose f_1, f_2 to obtain any two prescribed eigenvalues in $\mathbb{C}$

Observation: there is no way to control the system by just observing the position: $u(t) = f_1\,\vartheta(t)$
$(f_2 = 0)$: for any choice of f_1, the matrix

$$A+BF = \begin{bmatrix} 0 & 1 \\ g+f_1 & 0 \end{bmatrix}$$

will never have $\Lambda(A+BF) \subset LHP$

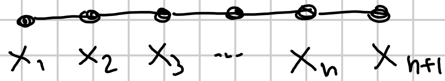
Example: controlling heating.
Let us consider a long corridor of length 1



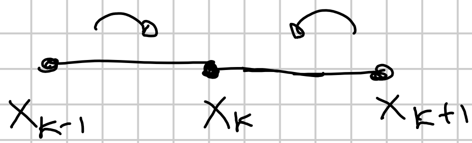
Let $x(y,t)$ = temperature at point $y \in [0,1]$ and time t.

$$\frac{\partial}{\partial t} x(y,t) = \alpha \frac{\partial^2}{\partial y^2} x(y,t) \qquad \alpha \text{ constant } > 0$$

Discretization: Let us divide the corridor into n equal intervals

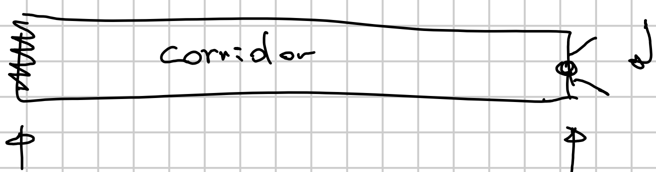


The variation in temperature in point x_i depends on the difference of temperature with its neighbors



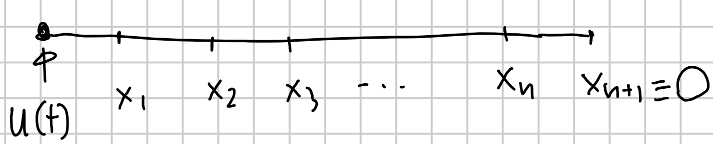
$$\frac{\partial}{\partial t} x_k(t) = \alpha \frac{\left(x_{k+1}(t) - x_k(t)\right) + \left(x_{k-1}(t) - x_k(t)\right)}{h^2}$$

open window: $x_{n+1}(t) \equiv 0$



heating element:
control $u(t)$

$$\frac{\partial}{\partial t}\begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} = \underbrace{\frac{\alpha}{n^2}\begin{bmatrix} -2 & 1 & & & 0 \\ 1 & \ddots & \ddots & & \\ & 1 & -2 & 1 & \\ & & 1 & -2 & 1 \\ 0 & & & \ddots & \ddots \\ & & & 1 & -2 \end{bmatrix}}_{A} \cdot \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{B} u(t)$$



$$\boxed{\frac{\partial}{\partial t} x(t) = A x(t) + B u(t)}$$

$x : [0, \infty) \to \mathbb{R}^n$

$u : [0, \infty) \to \mathbb{R}$