Control theory

$$\frac{d}{dt} x(t) = A x(t) + B u(t)$$

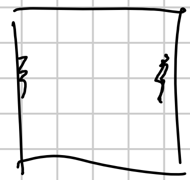
$$x(0) = x_0$$

$A \in \mathbb{C}^{n \times n}$ $B \in \mathbb{C}^{n \times m}$

control function

$u : [0, \infty) \to \mathbb{R}^m$

($u \in C^\infty$, for instance)



$x \equiv 0$ $u \equiv 0$ is an equilibrium point

Q_1. Can we <u>stabilize</u> the system, i.e. choose $u(t)$ s.t. $\lim_{t \to \infty} x(t) = 0$ (for all x_0)

Q_2. Can we <u>control</u> the system, i.e., given $x_F \in \mathbb{C}^n$, $t_F > 0$, choose $u(t)$ s.t. $x(t_F) = x_F$

e.g. <u>inverted pendulum</u> $A = \begin{bmatrix} 0 & 1 \\ g & 0 \end{bmatrix}$ $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ feedback control

One can find $F \in \mathbb{C}^{1 \times 2}$ s.t. $u(t) = F x(t)$ stabilizes the system, because $\Lambda(A+BF) \subset LHP$ and then

$$\dot{x} = Ax + Bu = (A+BF)x$$ is asympt. stable

closed-loop system

<u>Example</u>: if

$$A = \begin{array}{cc} & \begin{array}{cc} n_1 & n_2 \end{array} \\ \begin{array}{c} \\ \\ \end{array} & \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \begin{array}{c} n_1 \\ n_2 \end{array} \end{array} \qquad B = \begin{array}{c} m \\ \begin{bmatrix} B_1 \\ 0 \end{bmatrix} \begin{array}{c} n_1 \\ n_2 \end{array} \end{array},$$

then $x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \begin{array}{c} n_1 \\ n_2 \end{array}$

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u(t) = \begin{bmatrix} A_{11}x_1(t) + A_{12}x_2(t) + B_1 u(t) \\ A_{22}x_2(t) \end{bmatrix}$$

no control here

Second block: $\frac{d}{dt} x_2(t) = A_{22} x_2(t)$

$$x_2(t) = \exp(t A_{22}) x_{02}$$

No possibility to change it; if $\Lambda(A_{22}) \not\subset LHP$, this part will <u>not</u> be stable.

Essentially, this is the only obstruction, but it might be hidden behind a change of basis for x:

$$A = M\begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} M^{-1}, \quad B = M\begin{bmatrix} B_1 \\ 0 \end{bmatrix} \qquad M \in \mathbb{C}^{n\times n} \text{ invert.}$$

$$\hat{x} = Mx$$

Given (A,B), how do we identify if we are in this case?

<u>Invariant subspace</u>: $\operatorname{Im}\begin{bmatrix} I \\ 0 \end{bmatrix}$ is an invariant subspace for A, and it contains the columns of B.

$\operatorname{Im} M\begin{bmatrix} I \\ 0 \end{bmatrix}$ = first n_1 columns of M is an inv. sub. for

$M\begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} M^{-1}$, and it contains columns of $M\begin{bmatrix} B_1 \\ 0 \end{bmatrix}$

Q: Given A, B, can we determine if there is a nontrivial $\neq \mathbb{C}^n$ invariant subspace for A that contains $\operatorname{Im} B$?

<u>Lemma</u>: Given $A \in \mathbb{C}^{n\times n}$, $B \in \mathbb{C}^{n\times m}$, the smallest A-invariant subspace that contains the columns of B is

$$K(A,B) = \operatorname{Im}\begin{bmatrix} B & AB & A^2B & A^3B & \cdots \end{bmatrix}$$

<u>Proof</u>: 1) $K(A,B)$ is A-invariant: $A\begin{bmatrix} B & AB & A^2B & \cdots \end{bmatrix} = \begin{bmatrix} AB & A^2B & A^3B & \cdots \end{bmatrix}$

Any lin. comb. of the columns of $[B \ AB \ A^2B \ \cdots]$, multiplied by A, gives another lin. comb.

2) Any A-invariant subspace that contains the columns of B must contains the columns of $B, AB, A^2B, A^3B, \ldots$ $\square$

This resembles $K_n(A,b) = \operatorname{Im}[b \ Ab \ \cdots \ A^{n-1}b]$

<u>Remark</u>: $\operatorname{Im}[B \ AB \ A^2B \ A^3B \ \cdots] = \operatorname{Im}[B \ AB \ A^2B \ \cdots \ A^{n-1}B]$

because A^n is a linear comb. of $A^{n-1}, A^{n-2}, \ldots A^1, I$

(Cayley-Hamilton theorem: $p(A)=0$, where $p(\lambda) = \det(A-\lambda I)$, $\deg p = n$)

<u>Def</u>: $K(A,B) = \operatorname{Im}[B \ AB \ A^2B \ \cdots]$ is called <u>controllable space</u> of (A,B), and (A,B) is called <u>controllable</u> if $K(A,B) = \mathbb{C}^n$

<u>Lemma</u>: (Kalman decomposition): for any A, B, there exists a nonsingular $M \in \mathbb{C}^{n \times n}$ such that

$$A = M \begin{array}{c} \begin{array}{cc} n_1 & n_2 \end{array} \\ \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \end{array} M^{-1}, \quad B = M \begin{bmatrix} B_1 \\ 0 \end{bmatrix} \begin{array}{c} n_1 \\ n_2 \end{array}$$

$n_1 = \dim K(A,B)$, and (A_{11}, B_1) is controllable.

(If (A,B) is already controllable, $n_1 = n$ and the second block does not exist)

<u>Proof</u>: Take M s.t. $M = [M_1 \ M_2]$, and the columns of M_1 are a basis of $K(A,B)$. Then, $A = M \begin{bmatrix} * & * \\ 0 & * \end{bmatrix} M^{-1}$ because M_1 is A-invariant, and $B = M \begin{bmatrix} * \\ 0 \end{bmatrix}$ because $\operatorname{Im} B \subset \operatorname{Im} M_1$.

(A_{11}, B_1) is controllable, because otherwise one would have a smaller invariant subspace $\operatorname{Im} N_1$ A_{11}-invariant and that contains the columns of B, and then $M \cap N$, would be a smaller inv. subspace for A that contains the columns of B.

Theorem: the following are equivalent:

1) The system $\dot{x} = Ax + Bu$ is controllable, i.e. for each x_F, t_F there is $u(t)$ s.t. $x(t_F) = x_F$ $\qquad t_F > 0$

2) (A,B) is controllable: $\operatorname{Im}[B \; AB \; A^2B \cdots] = \mathbb{C}^n$

3) The matrix

$$W_t = \int_0^t \exp(A\tau) B B^* \exp(A^* \tau) \, d\tau$$

is invertible (for a specific $t>0$, or for all $t>0$ equivalently).

Proof: $1 \Rightarrow 2$: (not-2 $\Rightarrow$ not-1)

Recall the formula for sol. $\dot{x}(t) = Ax(t) + \underbrace{f(t)}_{=Bu(t)}$

$$x(t) = \exp(At) x_0 + \int_0^t \underline{\exp(A(t-\tau))} B u(\tau) \, d\tau$$

Suppose $K(A,B) \neq \mathbb{C}^n$. Then, since $\exp(A(t-\tau))$ is a polynomial in A, the integral takes values in $K(A,B) \subsetneq \mathbb{C}^n$

Hence $x(t_F) - \exp(At_F)x_0 \in K(A,B)$ is not the whole $\mathbb{C}^n$, and the system is not controllable.

$2 \Rightarrow 3$ (not-3 $\Rightarrow$ not 2)

Suppose $W_t v = 0$ for some $v \neq 0$. Then,

$$0 = v^* W_t v = \int_0^t \underbrace{v^* \exp(A\tau) B}\, \underbrace{B^* \exp(A^*\tau) v} \, d\tau$$

$$= \int_0^t \|\varphi(\tau)\|^2 d\tau, \quad \text{where } \varphi(\tau) = v^* \exp(A\tau) B$$

Since $\varphi(\tau)$ is continuous, it must be the case that $\varphi(\tau) \equiv 0$

In particular,

$$0 = \varphi(0) = v^* B = 0$$
$$0 = \varphi'(0) = v^* AB = 0$$
$$0 = \varphi''(0) = v^* A^2 B = 0$$

Hence $v^* [B \; AB \; A^2B \; \cdots] = 0$. v is orthogonal to $K(A,B)$, thus $K(A,B) \neq \mathbb{C}^n$.

$3 \Rightarrow 1$ Let us choose the control function of the form

$$u(t) = B^* \exp(A^*(t_F - t)) y \qquad \text{for a certain } y \in \mathbb{C}^n.$$

$$x(t_F) = \exp(At_F) x_0 + \int_0^{t_F} \exp(A(t_F - \tau)) B \underbrace{B^* \exp(A^*(t_F - \tau)) y}_{u(t)} \, d\tau$$

$$= \exp(At_F) x_0 + W_{t_F} \cdot y$$

So, given x_F, I can choose y s.t.

$$x_F - \exp(At_F) x_0 = W_{t_F} \cdot y$$

and obtain $x(t_F) = x_F$. $\square$

One more controllability condition:

Thm (Popov, or Hautus condition)

(A,B) controllable $\iff$ rank $[A - \lambda I \;\; B] = n$ for all $\lambda \in \mathbb{C}$

(Note that if $\lambda \notin \Lambda(A)$, then rank $A - \lambda I = n$, hence rank $[A - \lambda I \;\; B] = n$)

Proof: $\Rightarrow$) by contradiction: assume $v^*\begin{bmatrix} A-\lambda I & B\end{bmatrix}=0$
for some $v\neq 0$, $\lambda\in\mathbb{C}$. Then, one sees that
$$0=v^*B=v^*AB=v^*A^2B=\dots$$

$\Leftarrow$) by contradiction: assume that (A,B) can be written (up to a change of basis)
$$A=\begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22}\end{bmatrix}, \quad B=\begin{bmatrix} B_1 \\ 0\end{bmatrix}.$$
Then, take a left eigenpair $w^*(A_{22}-\lambda I_{n_2})=0$
then
$$\begin{bmatrix} 0 & w^*\end{bmatrix}\begin{bmatrix} A-\lambda I & B\end{bmatrix}=0.$$

How to test controllability in practice?

1) compute $\mathrm{svd}\left(\begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B\end{bmatrix}\right)$
2) compute W_∞ as the solution of $AW+WA^*+BB^*=0$
3) Compute eigenvalues of A, and for each of them check that $\mathrm{rank}\begin{bmatrix} A-\lambda I & B\end{bmatrix}=n$

Matlab experiment: pair close to uncontrollable.

$$A=\begin{bmatrix} 0 & 1 \\ 9.8 & 0\end{bmatrix} \quad A+Bf=\begin{bmatrix} 0 & 1 \\ 9.8+f_1 & f_2\end{bmatrix}$$