Stabilizability of a control system

$$\dot{x}(t) = Ax(t) + Bu(t) \qquad A \in \mathbb{C}^{n\times n},\ B \in \mathbb{C}^{n\times m}$$
$$x(0) = x_0$$

<u>Def</u>: the system is <u>stabilizable</u> if $\exists\ u : [0,\infty) \to \mathbb{C}^m$ s.t. $\lim_{t\to\infty} x(t) = 0$

<u>Ex</u>: $A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$ $B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}$ $\Leftrightarrow \begin{cases} \dot{x}_1 = A_{11}x_1 + A_{12}x_2 + B_1 u \\ \dot{x}_2 = A_{22}x_2 \end{cases}$

If A_{22} is a stable matrix, i.e. $\Lambda(A_{22}) \subset LHP$, then the system is not controllable, but it is stabilizable: $\lim_{t\to\infty} x_2(t) = 0$ already, and one can control the first part (if (A_{11}, B_1) controllable)

Recall: <u>Kalman decomposition</u>: every pair (A,B) can be written up to a change of basis as $A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$, $B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}$, with (A_{11}, B_1) controllable (and $n_2 = 0$ if necessary)

<u>Theorem</u>: (stabilizability conditions)

The following are equivalent:

1. The system $\dot{x} = Ax + Bu$ is stabilizable
2. In the Kalman decomposition, $\Lambda(A_{22}) \subset LHP$
3. $\operatorname{rk} \begin{bmatrix} A - \lambda I & B \end{bmatrix} = n$ for all $\lambda \in \mathbb{C} \setminus LHP$

4. There exists $F \in \mathbb{C}^{m \times n}$ s.t. $u(t) = Fx(t)$ is a stabilizing control, i.e., $\Lambda(A+BF) \subset LHP$ and $\lim x(t) = 0$.

Remark: when (A,B) stabilizable, there are an infinite number of stabilizing feedback matrices F, by continuity (if $\Lambda(A+BF) \subset LHP$, the same holds for small perturbations to F)

This means that we can impose more conditions.

Optimal control (linear-quadratic regulator)

Find $u : [0,\infty) \to \mathbb{C}^m$ that minimizes

$$V(u) = \int_0^\infty x(t)^* Q\, x(t) + u(t)^* R\, u(t)\, dt$$

for given $Q \in \mathbb{C}^{n \times n}$, $Q = Q^* \succeq 0$, $R \in \mathbb{C}^{m \times m}$, $R = R^* \succeq 0$.
s.t. $\dot{x} = Ax + Bu$

Idea: control reducing the oscillations and the "input" u.

We shall assume $R \succ 0$

Theorem:

Let $Q \succeq 0$, $R \succ 0$, (A,B) stabilizable, (A^*, Q) stabilizable. $x_0 \in \mathbb{C}^n$ be given

Then, there exists $X \in \mathbb{C}^{n \times n}$, $X = X^* \succeq 0$ s.t.

1. $A^* X + XA + Q - XGX = 0$, $\quad G = BR^{-1}B^* \in \mathbb{C}^{n \times n}$, $G = G^* \succeq 0$
2. $\Lambda(A - GX) \subset LHP$.

Then, the optimum of the minimum problem

$$\min \int_0^\infty x(t)^* Q x(t) + u(t)^* R u(t) \, dt$$

$$\text{s.t.} \begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ x(0) = x_0 \in \mathbb{C}^n \end{cases} \qquad \lim_{t\to\infty} x(t) = 0$$

is $x_0^* X x_0$, attained with $u(t) = Fx(t)$
with $F = -R^{-1}B^* X$.

The matrix eqn in 1. is called algebraic Riccati equation
X that satisfies 2. (in addition to 1.) is called a stabilizing solution.

Note that $A + BF = A + B(-R^{-1}B^* X) = A - \underbrace{BR^{-1}B^*}_{= G} X = A - GX$.

Proof: We shall prove later the existence of X that solves 1. + 2., for now let us just give the connection to the minimum problem.
Let us take an arbitrary function u that is a stabilizing control, i.e., $\lim_{t\to\infty} x(t) = 0$. We compute

$$\frac{d}{dt} x(t)^* X x(t) = \dot{x}^* X x + x^* X \dot{x} = (Ax + Bu)^* X x +$$

$$+ x^* X (Ax + Bu) =$$

$$= x^* (\underbrace{A^* X + XA}_{= XBR^{-1}B^*X - Q}) x + u^* B X x + x^* X B u$$

$$= x^* XBR^{-1}B^* X x - x^* Q x + u^* B X x + x^* X B u \quad + u^* R u - u^* R u$$

$$= \underbrace{(u + R^{-1}B^* X x)^* R (u + R^{-1}B^* X x)}_{\geq 0} \underbrace{- x^* Q x - u^* R u}_{\text{quantity under the integral}}$$

Hence,

$$\int_0^\infty x^*Qx+u^*Ru\,dt = \underbrace{\int_0^\infty (u+R^{-1}B^*Xx)^*R(u+R^{-1}B^*Xx)\,dt}_{\geq 0} - \underbrace{x^*Xx\Big|_0^\infty}_{\cancel{x(\infty)^*Xx(\infty)} - x_0^*Xx_0}$$

$x(\infty)^*Xx(\infty) = 0$

$$= x_0^*Xx_0 + (\text{something} \geq 0)$$

So the minimum is $x_0^*Xx_0$, attained with $u(t) = \underbrace{-R^{-1}B^*X}_{=F}x(t)$ $\forall t$

The most interesting part for us: solving

$$A^*X+XA+Q-XGX=0, \qquad \Lambda(A-GX) \subset LHP$$

Ex: in dimension $n=1$:

$$-gx^2+(\bar{a}+a)x+q=0 \qquad x_\pm = \frac{-(\bar{a}+a) \pm \sqrt{(\bar{a}+a)^2+4gq}}{-2g}$$

Since $g,q \geq 0$, the discriminant is real, and this has two real solutions, one positive and one negative, and $a-gx$ is negative in one case and positive in the other. (or LHP/RHP if $a \notin \mathbb{R}$)

Let us assume $a \in \mathbb{R}$, then

$$x_\pm = \frac{-a \pm \sqrt{a^2+gq}}{-g} \quad \begin{cases} \dfrac{-a+\sqrt{a^2+gq}}{-g} < 0 & \text{if } g>0,\ q>0 \\[2ex] \dfrac{-a-\sqrt{a^2-gq}}{-g} > 0 \end{cases}$$

$$a-gx_\pm = \pm\sqrt{a^2-gq}$$

The solution x_- has $\boxed{a-gx_- \in LHP, \quad x_- > 0}$

Linear algebra of the algebraic Riccati equation:

if $A^*X+XA+Q-XGX=0$, $\Lambda(A-GX) \subset LHP$, then

$$\begin{bmatrix} A & -G \\ -Q & -A^* \end{bmatrix} \begin{bmatrix} I \\ X \end{bmatrix} = \begin{bmatrix} I \\ X \end{bmatrix}(A-GX). \iff \begin{cases} A-GX = A-GX \\ -Q-A^*X = X(A-GX) \end{cases}$$

I.e., $\begin{bmatrix} I \\ X \end{bmatrix}$ is an invariant subspace for $\begin{bmatrix} A & -G \\ -Q & -A^* \end{bmatrix} =: \mathcal{H}$

(Hamiltonian matrix), and all the associated eigenvalues are in the LHP

We start by proving some properties of $\mathcal{H}$

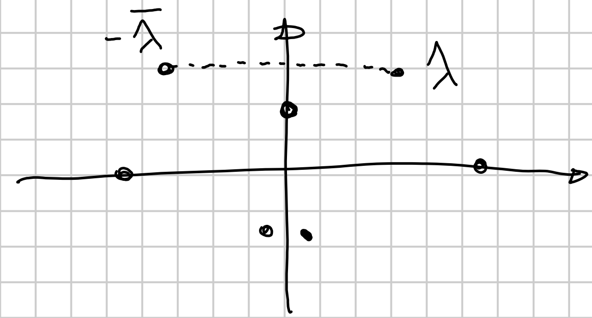
Lemma: let λ be an eigenvalue of $\mathcal{H}$, then $-\bar{\lambda}$ is also an eigenvalue of $\mathcal{H}$, with the same alg. multiplicity.

Proof: We shall prove that $\mathcal{H}$ is similar to $-\mathcal{H}^*$

Let $J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Then, we can compute

$$J^{-1} \mathcal{H} J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} A & -G \\ -Q & -A^* \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} Q & A^* \\ A & -G \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -A^* & Q \\ G & A \end{bmatrix}$$

$$= -\mathcal{H}^*.$$



$\lambda, -\bar{\lambda}$ are symmetric wrt the imaginary axis

The eigenvalues of $\mathcal{H}$ are symmetric wrt the im. axis.

We shall prove that there are no eigenvalues on the im. axis, hence $\mathcal{H} \in \mathbb{C}^{2n \times 2n}$ has n eigenvalues in the LHP and n in the RHP.

Theorem: Assume $Q \succcurlyeq 0$, $G = BR^{-1}B^* \succcurlyeq 0$,

(A, B), (A^*, Q) stabilizable.

$\Updownarrow$

(A, G) stabilizable $\qquad \left(K(A,B) = \operatorname{Im}[B \;\; AB \;\; A^2B \; \cdots] = \right.$

$\left. = K(A,G) = \operatorname{Im}[G \;\; AG \;\; A^2G \; \cdots] \right.$

because $\operatorname{Im} B = \operatorname{Im} G$)

Then, H has no purely imaginary eigenvalue.

<u>Proof</u>: assume, by contradiction, that $\lambda \in i\mathbb{R}$

$v \neq 0 \Rightarrow v_1 \neq 0$ or $v_2 \neq 0$

$$0 = (H - \lambda I)v = \begin{bmatrix} A-\lambda I & -G \\ -Q & -(A-\lambda I)^* \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \Longleftrightarrow \begin{cases} (A-\lambda I)v_1 - Gv_2 = 0 \\ -Qv_1 - (A-\lambda I)^* v_2 = 0 \end{cases}$$

Multiply the first eq. by v_2^*, and the second transposed by v_1

$$\begin{cases} v_2^*(A-\lambda I)v_1 - v_2^* G v_2 = 0 \\ -v_1^* Q v_1 - v_2^*(A-\lambda I)v_1 = 0 \end{cases} \rightarrow v_1^* Q v_1 + v_2^* G v_2 = 0$$

Since $Q \succcurlyeq 0$, $G \succcurlyeq 0$, it must be $Qv_1 = Gv_2 = 0$

and so $(A-\lambda I)v_1 = (A-\lambda I)^* v_2 = 0$

$$v_2^* \begin{bmatrix} A-\lambda I & G \end{bmatrix} = 0$$
$$v_1^* \begin{bmatrix} A^*+\lambda I & Q \end{bmatrix} = 0$$

Since one among v_1, v_2 must be nonzero,

we contradict stabilizability of (A, G) or (A^*, Q)

Now we can compute a basis for the inv. subspace associated to $\Lambda(H) \cap LHP$:

$$\begin{bmatrix} A & -G \\ -Q & -A^* \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} S \qquad \Lambda(S) \subset LHP$$

If U_1 is invertible, I can write

$$\begin{bmatrix} A & -G \\ -Q & -A^* \end{bmatrix} \underbrace{\begin{bmatrix} U_1 \\ U_2 \end{bmatrix} U_1^{-1}}_{\begin{bmatrix} I \\ X \end{bmatrix}} = \underbrace{\begin{bmatrix} U_1 \\ U_2 \end{bmatrix} U_1^{-1}}_{\begin{bmatrix} I \\ X \end{bmatrix}} \underbrace{U_1 S U_1^{-1}}$$

with $X = U_2 U_1^{-1}$

$\downarrow$

$\Lambda(U_1 S U_1^{-1}) \subset LHP$

Hence

$$\begin{bmatrix} A & -G \\ -Q & -A^* \end{bmatrix}\begin{bmatrix} I \\ X \end{bmatrix} = \begin{bmatrix} I \\ X \end{bmatrix}\hat{S} \iff \begin{cases} A - GX = \hat{S} \\ -Q - A^*X = X\hat{S} = X(A-GX) \end{cases}$$

$\Lambda(A-GX) \subset LHP$

$\leftarrow$ ARE

So $X = U_2 U_1^{-1}$ is a stabilizing solution of the ARE.

<u>Theorem</u>: under the same assumptions as the above results

$Q \succcurlyeq 0$, $G \succcurlyeq 0$, (A,B), (A^*,Q) stab.,

U_1 is invertible

<u>Proof</u>: omitted (see notes if you are curious).

Also, $X = X^* \succcurlyeq 0$ (also not proved here)

How to solve AREs?

1. Via Schur form + reordering + inv. subspace
2. Via $\operatorname{sign}(\mathcal{H})$
3. Newton's method
4. Structured eigenvalue methods