

Solving polynomial eigenvalue problems by means of the Ehrlich-Aberth method

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Abstract

Given the $n \times n$ matrix polynomial $P(x) = \sum_{i=0}^k P_i x^i$, we consider the associated polynomial eigenvalue problem. This problem, viewed in terms of computing the roots of the scalar polynomial $\det P(x)$, is treated in polynomial form rather than in matrix form by means of the Ehrlich-Aberth iteration. The main computational issues are discussed, namely, the choice of the starting approximations needed to start the Ehrlich-Aberth iteration, the computation of the Newton correction, the halting criterion, and the treatment of eigenvalues at infinity. We arrive at an effective implementation which provides more accurate approximations to the eigenvalues with respect to the methods based on the QZ algorithm. The case of polynomials having special structures, like palindromic, Hamiltonian, symplectic, etc., where the eigenvalues have special symmetries in the complex plane, is considered. A general way to adapt the Ehrlich-Aberth iteration to structured matrix polynomial is introduced. Numerical experiments which confirm the effectiveness of this approach are reported.

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1. Introduction

Given two positive integers k , n and matrices $P_j \in \mathbb{C}^{n \times n}$, $j = 0, \dots, k$ consider the matrix polynomial

$$P(x) = \sum_{j=0}^k P_j x^j \quad (1)$$

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where $P_k \neq 0$, so that $P(x)$ has degree k , and define the scalar polynomial $p(x) = \det P(x)$ of degree $N \leq nk$. Assume that $P(x)$ is regular, that is $p(x)$ is not identically zero.

Under such an assumption, the polynomial eigenvalue problem associated with $P(x)$ consists in computing the roots of the polynomial $p(x)$ which are called the eigenvalues of the matrix polynomial $P(x)$. Observe that, if P_k has not full rank, then $N < nk$. In this case, it is convenient to introduce $nk - N$ eigenvalues at infinity and say that the matrix polynomial $P(x)$ has nk eigenvalues including the $nk - N$ eigenvalues at infinity.

Our interest is addressed to the design and analysis of efficient algorithms for the polynomial eigenvalue problem based on the Ehrlich-Aberth iteration [1, 7, 9].

Recently, much literature has been addressed to the polynomial eigenvalue problem (PEP). For the numerical solution of PEPs, fast and numerically stable methods are sought. Several algorithms have been introduced based on the technique of linearization where the polynomial problem is replaced by a linear pencil with larger size and the customary methods for the generalised eigenvalue problem are applied. For more details, see for instance [18, 27, 28, 47] and the references therein.

Specific attention concerns structured problems where the matrix coefficients P_j have some additional property which is reflected on structural properties of the roots. For instance, in the case of T-palindromic polynomials [23, 43], where $P_j = P_{k-j}^T \in \mathbb{C}^{n \times n}$, the roots are encountered in pairs $(x, 1/x)$. In general, we may consider the case of structures where the roots can be grouped in pairs as $(x, f(x))$, where $f(x)$ is any analytic function such that $f(x) = f^{-1}(x)$ [14]. In this case the goal is to design algorithms which take advantage of this additional information about the eigenvalues and deliver approximations to the eigenvalues which respect these symmetries independently of the rounding errors.

The Ehrlich-Aberth iteration (EAI) was historically first mentioned in [7] and afterwards independently rediscovered many times. It is one of the many simultaneous iteration techniques available in the literature for the numerical approximation of polynomial roots [29, 40]. In [3, 4] the EAI has been combined with various techniques like the Rouché theorem, the Newton polygon technique, and the Gerschgorin inclusion theorems for arriving at efficient and robust software implementations. The package Polzeros, designed in [3], provides a robust and reliable tool for approximating roots of polynomial in floating point arithmetic. The package MPSolve designed in [4] provides certified approximations to any desired number of digits of the roots of any polynomial.

The EAI has been used in [5] to solve the generalised tridiagonal eigenvalue problem where the software provides effective accelerations in terms of CPU time. It has been used in [41] for quadratic hyperbolic tridiagonal eigenvalue problems.

In this paper we present an adaptation of the Ehrlich-Aberth method for the numerical solution of PEPs. The main computational issues that we analyze are the choice of the starting approximations, the computation of the Newton correction, the halting criterion, the design of *a posteriori* error bounds, and the

management of the problematic (multiple) eigenvalues at zero and at infinity.

Concerning the choice of the starting approximations, we propose a generalization to matrix polynomials of a technique introduced in [3] for scalar polynomials based on the Rouché theorem. In fact, we rely on the generalization to matrix polynomials of the Rouché theorem given in [34] and provide a way to determine an annulus in the complex plane which contains all the eigenvalues.

The Newton correction is computed by means of Jacobi's formula for the differential of the determinant of any square matrix in terms of the trace of the matrix $P(z)^{-1}P'(z)$. The halting condition is given in terms of the condition number of $P(z)$. Eigenvalues at zero and at infinity can be removed, to a certain extent, by using the specific features of the EAI relying on the information provided by the singular values of P_0 and P_k . A posteriori error bounds are given by constructing a set of inclusion disks relying on the Gerschgorin theorem and adapting the results of [45] to the case of matrix polynomials.

The computational analysis of this method shows that the number of arithmetic operations (ops) is $O(k^2n^3 + kn^4)$. In the case where the degree k is large with respect to the square root of the matrix size n , this complexity bound compares favourably with the bound $O(k^3n^3)$ of the customary matrix-based algorithms like the QZ applied to a linearization. Cases of this kind can be encountered, for instance, in the truncation of matrix power series [49].

The EAI does not compute the eigenvectors, which are sometimes needed as well. Nevertheless, after a good approximation of the eigenvalues is obtained, other methods, e.g. the SVD or the inverse iteration, can be used to compute the eigenvectors without increasing the complexity of the algorithm. We were able to compute eigenvectors with high accuracy using the eigenvalues given by the EAI.

We consider the case of polynomials endowed with specific properties like palindromic, T-palindromic, Hamiltonian, symplectic polynomials, whose eigenvalues have special symmetries in the complex plane. We propose a unifying treatment of this class of structured polynomials and show how the EAI can be adapted to deal with these classes in a very effective way. In fact, our variant of the EAI enables one to compute only a subset of eigenvalues and to recover the remaining part of the spectrum by means of the symmetries satisfied by the eigenvalues. By exploiting the structure of the problem, this approach leads to a saving on the number of operations and provides algorithms which yield numerical approximations fulfilling the symmetry properties.

We conclude our discussion by presenting the results of several numerical experiments performed in order to test the effectiveness of our approach in terms of speed and of accuracy. We have compared the Ehrlich-Aberth iteration with the Matlab functions `polyeig` and `quadeig`[16]. In the structured case, we have also considered, when available, other structured methods, say, the URV algorithm by Schröder [43].

We show that the EAI is much faster than the available techniques in the case where the degree is larger with respect to the size of the matrices. Moreover, for the test problems NLEVP of [2], it turns out that the accuracy of the computed approximations is generally better than the accuracy obtained with the available

algorithms.

The paper is organised as follows. In Section 2 we recall the Ehrlich-Aberth method and discuss the main computational issues encountered in its implementation. In Section 3 we consider the case of “structured polynomials”, i.e. the case where the matrix coefficients have some special properties. Section 4 reports the results of the numerical experiments.

2. The Ehrlich-Aberth method for matrix polynomials

Given a vector $y^{(0)} \in \mathbb{C}^N$ of initial guesses for the N roots of the polynomial $p(x)$, the EAI provides the sequence of simultaneous approximations $y^{(i)}$ given by

$$y_j^{(i+1)} = y_j^{(i)} - \frac{\mathcal{N}(y_j^{(i)})}{1 - \mathcal{N}(y_j^{(i)})\mathcal{A}_j(y^{(i)})}, \quad \mathcal{N}(x) = \frac{p(x)}{p'(x)} \quad (2)$$

$$\mathcal{A}_j(y^{(i)}) = \sum_{\ell=1, \ell \neq j}^N \frac{1}{y_j^{(i)} - y_\ell^{(i)}}$$

where $\mathcal{N}(x)$ is the Newton correction. It is easy to check that the j th update in (2) is nothing else but the Newton iteration applied to the rational function $p(x)/\prod_{\ell=1, \ell \neq j}^N (x - y_\ell^{(i)})$, so that the EAI provides a way to implement the implicit deflation of the roots.

Besides the Jacobi-style version of EAI we may formulate the Gauss-Seidel version of EAI, that is

$$y_j^{(i+1)} = y_j^{(i)} - \frac{\mathcal{N}(p(y_j^{(i)}))}{1 - \mathcal{N}(p(y_j^{(i)}))\mathcal{A}_j(y^{(i)}, y^{(i+1)})}, \quad (3)$$

$$\mathcal{A}_j(y^{(i)}, y^{(i+1)}) = \sum_{\ell=1}^{j-1} \frac{1}{y_j^{(i)} - y_\ell^{(i+1)}} + \sum_{\ell=j+1}^N \frac{1}{y_j^{(i)} - y_\ell^{(i)}}.$$

The method, in the Jacobi version, is known to converge cubically for simple roots and linearly for multiple roots [40]. In the Gauss-Seidel version, convergence is slightly faster. In practice, good global convergence properties are observed; a theoretical analysis of global convergence, though, is still missing and constitutes an open problem.

With the term *vector iteration* of the Ehrlich-Aberth method we refer to the step which provides the vector $y^{(i+1)}$ given the vector $y^{(i)}$. We use the term *scalar iteration* for indicating the single step performed on the generic scalar component of the vector $y^{(i)}$.

In the case of a scalar polynomial of degree N the cost of a scalar iteration is $O(N)$ arithmetic operations. In this way, the cost of a vector iteration is at most $O(N^2)$ ops and is substantially reduced when most components have been numerically approximated, so that few scalar iterations must be performed in order to carry out the vector iteration.

The number of scalar iterations needed by the floating point implementation in order to find approximations which are exact roots of a slightly perturbed polynomial is, in practice, $O(N)$ if the starting approximations are computed by means of the Newton polygon technique [3]. This technique is particularly effective when the polynomial has roots with moduli which are very unbalanced.

Crucial aspects for an effective implementation of the EAI to matrix polynomials are

1. the computation of the Newton correction $p(x)/p'(x)$ given the value of x and of the input coefficients P_j , $j = 0, \dots, k$;
2. a criterion for stopping the iterations;
3. the choice of the initial approximations.

2.1. Computing the Newton correction

In the literature, methods based on some factorizations of $P(x)$ were developed to compute the Newton correction for functions that have the same zeros of $p(x)$: e.g., the method in [21], later proved to lack theoretical rigour and corrected in [19]. Other kinds of Newton-like approaches were presented in [20, 42].

If one wishes to work with $p(x)$ itself, a naive way to compute the Newton correction $p(x)/p'(x)$ would be to evaluate first the coefficients of the polynomial $p(x)$, say, by means of the evaluation-interpolation technique, and then to apply right after the Ehrlich-Aberth method to the scalar equation $p(x) = 0$. This approach would however come across numerical problems due to numerical instability and to overflow and underflow situations encountered in the computation of determinants.

It is therefore wise to conceive a strategy to avoid the explicit calculation of the coefficients of $p(x)$.

An effective way rests upon the well-known Jacobi's formula for the differential of the determinant of any invertible square matrix A :

$$d(\ln(\det A)) = \text{tr}(A^{-1}dA), \quad (4)$$

where tr denotes the trace. This way, we obtain the following expression for the derivative $p'(x)$

$$p'(x) = \frac{d \det(P(x))}{dx} = \det(P(x)) \cdot \text{tr}(P(x)^{-1} \cdot \frac{dP(x)}{dx}).$$

This formula allows us to evaluate the Newton correction $p(x)/p'(x)$, which is the centerpiece for the EAI, without explicitly calculating $p(x)$:

$$p(x)/p'(x) = \frac{1}{\text{tr}(P(x)^{-1}P'(x))}. \quad (5)$$

An evaluation of $P(x)$ and $P'(x)$ by means of Horner's method, followed by a numerical matrix inversion, allows to compute the trace of $P(x)^{-1}P'(x)$ in $\mathcal{O}(kn^2 + n^3)$ operations.

We mention that, even though we independently formulated it, we found out later on that the possible use of (4) in a numerical method for PEPs had been already suggested in [12, 22]. However, in these instances it was proposed to use it to apply the Newton method to approximate each single eigenvalue in sequence, mentioning the possibility to use an implicit deflation of previously found roots [30], in order to avoid that the method converges twice to the same eigenvalue. This leads to a formula akin to (2), (3), with the difference that the summation in the term \mathcal{A} is performed only up to the number of roots that have already been approximated. This is a crucial detail, because such a sequential implementation of the Newton method does not seem to achieve the same efficiency with respect to the EAI.

2.2. Stopping criterion

At the generic i th vector iteration it is crucial to decide whether the update of the j th component of the vector $y^{(i+1)}$ must be performed or the scalar iteration in that component must be halted.

Observe that, if ξ is a root of $p(x)$, that is $\det P(\xi) = 0$, then as the approximation x gets close to ξ the matrix $P(x)$ becomes ill-conditioned. This makes quite natural to stop the iterations if the reciprocal of the condition number $\mu(P(x))$ is less than a prescribed tolerance τ_1 . This criterion makes sense if the eigenvalue that we want to approximate is semi-simple. In the case of a defective eigenvalue λ with Jordan chains of length at most m , in view of the results in [39], it is more convenient to stop the iterations if the reciprocal of $\mu(P(x))$ is less than τ_1^m . This latter condition is hard to implement since it is not easy to evaluate numerically the length of the Jordan chains of a matrix polynomial. Using the former stopping criterion may lead to a premature halt of the algorithm in the case of defective eigenvalues.

As an alternative to the previous stopping condition, following [46], define $\alpha(x) = \sum_{\ell=0}^k |x^\ell|$. If $y_j^{(i)}$ is not an eigenvalue of $P(x)$ then the quantity

$$\eta(y_j^{(i)}) = \left(\left\| \left(P(y_j^{(i)}) \right)^{-1} \right\|_2 (1 + \alpha(y_j^{(i)})) \right)^{-1}$$

measures the backward error for the approximation $y_j^{(i)}$, and can be cheaply evaluated during the EAI. The iteration can be halted when $\eta(y_j^{(i)})$ is smaller than a given tolerance.

For simple eigenvalues, no significant differences emerged between the two alternative possibilities. Therefore, our default choice was in favour of the criterion based on the condition number.

It is also convenient to add, with the “or” logic operator, the following condition

$$|\mathcal{N}(y_j^{(i)}) / (1 - \mathcal{N}(y_j^{(i)}) \mathcal{A}_j(y^{(i)}, y^{(i+1)}))| \leq \tau_2 |y_j^{(i)}| \quad (6)$$

where τ_2 is a given tolerance. This condition says that the computed correction is too tiny and would not change the significant digits of the current approximation.

2.3. Choosing initial approximations

As pointed out in [1, 3, 15], practically effective choices of initial approximations for the EAI are complex numbers equally displaced along circles. For instance, in [1] it is proposed to choose initial approximations displaced along a circle centered at the origin of sufficiently large radius so that it contains all the roots. In [15] the radius of the circle is suitably chosen. This strategy does not work effectively for polynomials having zeros with very large and with very small moduli. In [3] this drawback is overcome by considering different circles centered at the origin of suitable radii. The computation of these radii relies on the Rouché theorem.

Here we try to extend this technique to a certain extent. We recall that, according to the Rouché theorem, if $s(x)$ and $q(x)$ are two polynomials such that

$$|s(x)| > |q(x)|, \quad \text{for } |x| = r,$$

then $s(x)$ and $s(x) + q(x)$ have the same number of roots in the open disk $\{z \in \mathbb{C} : |z| < r\}$. Applying this property with $s(x) = x^m$ and $q(x) = p(x) - s(x)$, for $0 \leq m \leq N$, implies that if $r^m > \sum_{j=0, j \neq m}^N |a_j| r^j$ then the polynomial $p(x)$ has m roots in the open disk of center 0 and radius r . This property is at the basis of the criterion described in [3], based on the Newton polygon construction, for choosing initial approximations equidistributed along different circles centered in 0.

In order to extend this criterion to the case of matrix polynomials we need a generalisation of the Rouché theorem to matrix polynomials. We report the following result of [34] which we rephrase in a simpler way better suited for our problem.

Theorem 1. *Let $S(x)$ and $Q(x)$ be matrix polynomials and let r be a positive real. If $S(x)^* S(x) - Q(x)^* Q(x)$ is positive definite for $|x| = r$, then the polynomials $\det S(x)$ and $\det(S(x) + Q(x))$ have the same number of roots of modulus less than r .*

The following result is an immediate consequence of the above theorem applied to the polynomial $P(x)$ of (1) with $S(x) = x^m P_m$ and $Q(x) = \sum_{i=0, i \neq m}^k x^i P_i$.

Corollary 1. *Assume that*

$$P_m^* P_m r^{2m} - \left(\sum_{j=0, j \neq m}^k P_j^* \bar{x}^j \right) \left(\sum_{j=0, j \neq m}^k P_j x^j \right) \succ 0, \quad \text{for } |x| = r, \quad (7)$$

where $A \succ B$ means that $A - B$ is positive definite. Then the matrix polynomial $P(x)$ has mk eigenvalues in the open disk of center 0 and radius r .

Observe that if $\det P_m = 0$ then condition (7) cannot be verified. In fact, the vector v such that $P_m v = 0$ would be such that

$$v^* \left(\sum_{j=0, j \neq m}^k P_j^* \bar{x}^j \right) \left(\sum_{j=0, j \neq m}^k P_j x^j \right) v \leq 0$$

which is absurd.

In particular, if $\det P_k \neq 0$ the above corollary, applied with $m = k$, implies that all the eigenvalues of $P(z)$ are included in the disk of center 0 and radius r provided that

$$r^{2k} P_k^* P_k - \left(\sum_{j=0}^{k-1} P_j^* \bar{x}^j \right) \left(\sum_{j=0}^{k-1} P_j x^j \right) \succ 0, \quad \text{for } |x| = r. \quad (8)$$

Observe that the latter condition is implied by

$$r^{2k} P_k^* P_k \succ \sum_{j=0}^{k-1} r^j P_j^* P_j + I \sum_{j>i} \rho(|P_j|^* |P_i| + |P_i|^* |P_j|). \quad (9)$$

Similarly, applying Corollary 1 with $m = 0$ provides a disk where $P(x)$ has no eigenvalues.

As an example of application, consider the 5×5 quadratic matrix polynomial $P(x) = Ax^2 + Bx + A^T$, where B is the tridiagonal matrix defined by the entries $[1, 2, 1]$, and A is the matrix with diagonal entries $100, 1, 1/1000, 1/100000$, superdiagonal entries equal to 1 and with zero entries elsewhere. The eigenvalues of $P(x)$ have moduli $2.0050\text{e}+05$, $1.4969\text{e}+03$, $1.0000\text{e}+00$, $1.0000\text{e}+00$, $1.0000\text{e}+00$, $1.0000\text{e}+00$, $6.6805\text{e}-04$, $4.9874\text{e}-06$. The criterion based on the above corollary in the form (7) yields the bound $4.4\text{e} - 6 < |x| < 2.24\text{e}5$ which is quite good. Applying condition (9) yields the bounds $1.96\text{e} - 6 < |x| < 5.1\text{e}5$ which is still good.

Similar results have been obtained in [44] in the framework of tropical algebras. A different heuristic approach, which relies on Corollary 1, is to select the values of the radii by considering the inequality $\|P_m\|_2 r^m > \sum_{i=0, i \neq m}^k \|P_i\|_2 r^i$ in place of (7). This strategy, applied in the form $r^k > \sum_{i=0}^{k-1} \|P_i\|_2 r^i$ leads to the criterion based on computing the Newton polygon of the polynomial $\sum_{i=0}^k \|P_i\|_2 x^i$. This is the default choice of the starting approximations performed in our implementations.

2.4. A posteriori error bounds

In the case of a scalar polynomial $p(x)$ of degree N , given a set of approximations x_1, \dots, x_N to the roots of $p(x)$ it is possible to prove that [45] the set of disks $D_i = D(x_i, r_i)$ of center x_i and radius $r_i = n|p(x_i)| / (p_N \prod_{j=1, j \neq i}^N (x_i - x_j))$ is such that

1. the union of the disks contains all the roots of $p(x)$
2. each connected component formed by the union of, say, c overlapping disks, contains c roots of $p(x)$.

The set formed by D_i , $i = 1, \dots, N$ with the above properties is said *set of inclusion disks*.

In the case of a matrix polynomial $P(x)$ where $\det P_k \neq 0$, it is quite cheap to compute a set of inclusion disks. In fact, if $P(x) = \Pi LU$ is the PLU factorization of $P(x)$, then $|p(x)| = |\det P(x)| = \prod_{j=1}^n |u_{j,j}|$, where $U = (u_{i,j})$.

Moreover, the leading coefficient p_N of $\det P(x)$ coincides with $\det P_k$ which can be computed once for all. Observe that the LAPACK routine `zgesv` which solves a linear system with the matrix $P(x)$, used to compute the Newton correction $1/\text{trace}(P(x)^{-1}P'(x))$, applied with $x = x_i$, provides at a negligible cost also the radius r_i .

The availability of a set of inclusion disks enables one to perform a cluster analysis. In fact, once an isolated disk has been detected, we have isolated a single eigenvalue of the matrix polynomial $P(x)$. Once we have detected a set of c overlapping disks isolated from the remaining inclusion disks, we have detected a cluster formed by c eigenvalues of $P(x)$.

A different *a posteriori* error bound can be obtained by using a classical result [17]. The disk of center x_i and radius $\hat{r}_i = n|p(x_i)/p'(x_i)|$ contains a root of the polynomial $p(x)$. However, the set of disks obtained in this way does not fulfill properties 1 and 2 of the set of inclusion disks. It is worth pointing out that the computation of \hat{r}_i is inexpensive since the Newton correction $p(x_i)/p'(x_i)$ is computed by the EAI. Moreover, this *a posteriori* error bound still holds if the leading coefficient P_k is singular.

2.5. Multiple eigenvalues

Computational difficulties may be encountered in the case of multiple eigenvalues. In fact, the rate of convergence for multiple eigenvalues is linear, with respect to the cubic behaviour for simple eigenvalues. Moreover, for defective eigenvalues the standard stop condition may lead to a premature halt. For this reason, if it is possible to detect *a priori* multiple eigenvalues, it is advisable to deflate them; if it is not possible to spot all of them theoretically, even lower bounds on the multiplicity are very helpful. If multiple eigenvalues are not predicted theoretically, one must rely on the cluster analysis to identify them and modify accordingly the stopping criterion.

A common situation that leads to multiple eigenvalues is met when the extremal coefficients are rank-deficient matrices. In this case, 0 and/or ∞ have multiplicity greater than or equal to 1. This situation can be circumvented to a certain extent.

In the case of m eigenvalues at infinity, one may just start with an approximation vector y of length $nk - m$, acknowledging that the determinant $p(x)$ has in fact degree $nk - m$; if there are m zero eigenvalues it is possible to set to zero m components of the vector $y^{(0)}$ avoiding to update them.

The number of null singular values of P_0 provides a lower bound to the number of null eigenvalues of $P(x)$. Similarly, the number of zero singular values of P_k provides a bound to the number of eigenvalues at infinity. This way, the precomputation of the SVD of P_0 and P_k may increase the performance of the EAI. Equivalently, one may perform any rank-revealing factorization (e.g., QR) instead of the SVD. Sometimes, the structure of the coefficients allows to achieve better bounds (e.g., if the same rows/columns of many consecutive extremal coefficients are zero).

In our implementation, the rank of the extremal coefficients is tested. If it is less than n , it is also checked if P_0 and P_1 (resp., P_k and P_{k-1}) share any common

zero row/column. Thus, any manifest presence of zero and infinite eigenvalues is exploited, forcing deflation of all the guaranteed roots. Moreover, if the test detects the presence of eigenvalues at 0 (resp., ∞), in order to avoid a premature stop for other undetected eigenvalues at 0 (resp., ∞) the stopping criterion is made stricter. The stronger stop condition requires that, for eigenvalues smaller (resp., larger) than a given bound, either the relative correction criterion (6) is satisfied with tolerance τ_2 or the relative correction criterion (6) is satisfied with tolerance $\tau_2^{1/2}$ and, simultaneously, the reciprocal condition number criterion is satisfied with tolerance τ_1 . This heuristic device worked very effectively in our experiments, leading to satisfying results also in problems with multiple eigenvalues at either zero or infinity (see Section 4).

If the leading coefficient P_k is singular and if the degree of $p(x) = \det P(x)$ is not available together with the leading coefficients of $p(x)$, then it is not possible to generate a set of inclusion disks and to perform a cluster analysis. However, in this case we may apply an effective technique based on a rational transformation of the variable x . For instance, the variable x is replaced by the Möbius function $x = x(z) = (\alpha z + \beta)/(\gamma z + \delta)$ such that $\alpha\delta - \gamma\beta \neq 0$, and the polynomial $P(x)$ is replaced by the polynomial $Q(z) = (\gamma z + \delta)^k P(x(z))$. This way the eigenvalues at infinity of $P(x)$ are mapped into eigenvalues of $Q(z)$ at $-\delta/\gamma$. Moreover, $Q(z)$ has no eigenvalues at infinity provided that α/γ is not eigenvalue of $P(x)$. The substitution of variable can be performed implicitly without actually computing the coefficients of $Q(z)$ except for Q_k . We refer the reader to Section 3 and to [37] for more details.

2.6. Linearization as a possibility

In the paper [13] the computation of the Newton correction was carried out by first linearizing the polynomial by means of companion-like pencil, and then by evaluating the trace by means of an LQ factorization. This approach has a computational complexity of $\mathcal{O}(n^3 + kn^2)$ ops per scalar iteration, which is the same that is achieved by our algorithm. However, this cost can be reduced to $\mathcal{O}(n^2k^2)$ ops if the matrix pencil is reduced to triangular-Hessenberg form before the Jacobi formula is applied. This fact is the main advantage of using a linearization. However, linearization techniques, if not properly used, may lead to an undesired increasing of the eigenvalue condition numbers [18].

3. The case of structured polynomials

The EAI is particularly suited to deal with matrix polynomials endowed with specific structures of the matrix coefficients. We are interested in matrix structures which induce particular symmetries on the location of the eigenvalues. Polynomials of this kind are encountered in the applications and include, for instance, palindromic, T-palindromic, symplectic and Hamiltonian polynomials.

Customary PEP-solving algorithms, such as the application of the QZ to any suitable linearization of the polynomial, are not able to fully catch these symmetries of the spectrum. In the literature, there are specific matrix methods

that achieve this goal. The EAI enables to exploit the additional information both in the computation of the Newton correction and in the choice and in the management of the (initial) approximation of the roots in view of the structure-induced symmetries. We will see this later on.

Assume that the structured PEP is such that the eigenvalues appear in pairs $\{x, f(x)\}$, with $f(f(x)) = x \forall x$. A naive adaptation of the EAI to this property would be to apply (2) or (3) updating only the first half of the components of the vector y and simultaneously imposing $y^{(i)} = f(y^{(i-nk/2)})$, $i = nk/2 + 1, \dots, nk$. In numerical experiments, this approach seems not to be always efficient in terms of number of scalar iterations needed for numerical convergence. This motivates the design of more sophisticated structured variants of the EAI, that we are going to describe in the following.

Before analyzing the various classes of structured matrix polynomials, we recall some basic definitions of special matrices.

An $n \times n$ square matrix A is said to be *symmetric* if $A^T = A$ and *skew-symmetric* if $A^T = -A$. Let $n = 2m$. The matrix A is said to be *Hamiltonian* if it is such that $A^T J = -JA$ where J is the matrix $\begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix}$; A is said to be *skew-Hamiltonian* if it is such that $A^T J = JA$; A is said to be *symplectic* if it is such that $A^T J A = J$.

Every skew-Hamiltonian matrix can be obtained as the square of a Hamiltonian matrix, and conversely the square of a Hamiltonian matrix is always skew-Hamiltonian[11]. Symplectic matrices are exponential of Hamiltonian matrices.

3.1. Skew-Hamiltonian and even-dimensional skew-symmetric

A skew-symmetric polynomial $P(x)$ is a polynomial whose coefficients P_j , for $j = 0, \dots, k$, are skew-symmetric constant matrices. If the coefficients have even size $n = 2m$, we say that $P(x)$ is an even-dimensional skew-symmetric polynomial. A skew-Hamiltonian polynomial is defined as a polynomial whose coefficients P_j are all skew-Hamiltonian matrices. Classical eigenvalue problems for skew-Hamiltonian matrices [48] are a special case of skew-Hamiltonian PEPs.

These two classes of polynomials are closely related, because multiplication by J maps one class onto the other. A common feature is that the spectrum of any polynomial in these two classes contains only eigenvalues of even multiplicity. In fact, the determinant of a matrix polynomial $P(x)$ belonging to these two classes can be written as

$$p(x) = \det P(x) = q(x) \cdot q(x),$$

for a suitable polynomial $q(x)$. For the special case of a real skew-symmetric matrix pencil, a proof was given in [24] where a special Kronecker form was derived. The more general case comes from classical results on determinants [35]. Let us give here a simple proof of the statement for an even-dimensional skew-symmetric complex matrix polynomial using modern terminology.

Proposition 1. *Let $P(x) = -P(x)^T$ be a $2m \times 2m$ skew-symmetric matrix polynomial. Then $p(x) = \det P(x) = q(x) \cdot q(x)$ for some scalar polynomial $q(x)$.*

Proof. We shall prove the proposition by induction on m . For $m = 1$ the statement is obvious. Suppose now that any $(2m-2) \times (2m-2)$ skew-symmetric polynomial has the desired property. Let Π be a $2m \times 2m$ permutation matrix and let $Q(x) := \Pi P(x) \Pi^T$. Suppose that Π is such that

$$Q_0(x) := Q(1 : 2, 1 : 2) =: \begin{pmatrix} 0 & r(x) \\ -r(x) & 0 \end{pmatrix}$$

is nonsingular, where $r(x)$ is a suitable nonzero scalar polynomial. Notice that such an assumption can be safely made because if that was false for any Π then $P(x) = 0$ so $p(x) = 0$ and there would be nothing to prove. Now let $Q(x) = \begin{pmatrix} Q_0(x) & A(x) \\ -A(x)^T & Q_1(x) \end{pmatrix}$, where the polynomial matrices $A(x)$ and $Q_1(x)$ have, respectively, dimensions $2 \times (2m-2)$ and $(2m-2) \times (2m-2)$; also, let $\rho(x) := r(x)^{m-1}$. Define the rational function $S(x) := Q_1(x) + \frac{1}{r(x)} A(x)^T \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} A(x)$. Clearly, $r(x)S(x)$ is a $(2m-2) \times (2m-2)$ skew-symmetric matrix polynomial; therefore, by the inductive hypothesis, $\det S(x) = \frac{\theta(x)^2}{\rho(x)^2}$ where $\theta(x)$ is a suitable scalar polynomial. Moreover, $S(x)$ is the Schur complement of $Q_0(x)$. Thus, $p(x) = \frac{r(x)^2 \theta(x)^2}{\rho(x)^2}$, so $p(x)$ is the square of some scalar rational function $q(x) = \frac{\theta(x)}{r(x)^m}$. Since $p(x)$ is a polynomial, $q(x)$ must be a polynomial as well. \square

This is a particularly useful property which can be fully exploited by the Ehrlich-Aberth method. In fact, instead of applying the EAI to the polynomial $p(x)$ of degree $2mk$, one can apply the EAI to the polynomial $q(x)$ of degree mk even though $q(x)$ is not explicitly known.

More precisely, since $p'(x)/p(x) = 2q'(x)/q(x)$, one can compute the Newton correction $q(x)/q'(x)$ by means of

$$q(x)/q'(x) = 2p(x)/p'(x) = 2/\text{tr}(P(x)^{-1}P'(x)).$$

This way, the length of the vector of the approximations y in (2) or in (3) is reduced from $2mk$ to mk , moreover, the skew-Hamiltonian or the skew-symmetric structure of the coefficients can be exploited in the computation of $P(x)^{-1}P'(x)$.

3.2. Palindromic and symplectic

The polynomial $P(x)$ is called *purely palindromic* if $\text{Rev}P(x) = P(x)$, where the reversal polynomial $\text{Rev}(P(x))$ is defined by $\text{Rev}P(x) := x^k P(x^{-1})$. The polynomial $P(x)$ is called *T-palindromic* if $\text{Rev}P(x) = P(x)^T$. Both these structures induce a symmetry $(x, 1/x)$ in the spectrum. There is a vast literature on this kind of structure; see, e.g., [23, 25, 27, 43] and the references therein. The same structure appears in the standard eigenvalue problem for a symplectic matrix[10].

For this class of PEPs the change of variable $z := x + 1/x$ is useful. In [13] it was shown that the use of a non-standard polynomial basis, called the Dickson basis, leads to a suitable linearization of the purely palindromic case. Moreover, it was shown that if $P(x)$ is T-palindromic then it is possible to build a new skew-Hamiltonian matrix polynomial $M(z)$ such that $\det M(z(x)) = p(x) \cdot p(x)$; the Dickson basis was then used to obtain a useful linearization. In the following, we will show how to avoid the explicit use of the Dickson transformation. This has the advantage that there is no potential loss of accuracy for very large eigenvalues unlike the case of the algorithm in [13] where the linearization introduces unwanted defective eigenvalues at infinity which may create numerical problems (although such problems can be effectively amended by a structured refinement [14]).

If nk is even, then by means of simple formal manipulations one may show that $q(z) := x(z)^{-nk/2} \cdot p(x(z))$ is a polynomial in z , where $x(z) = (z + \sqrt{z^2 - 4})/2$ or $x(z) = (z - \sqrt{z^2 - 4})/2$, i.e., $z(x)$ is one of the two branches of the inverse function of $z(x) = x + 1/x$.

Moreover, taking derivatives in the latter equation leads to an explicit expression for the Newton correction $q(z)/q'(z)$ given in terms of $p(x)/p'(x)$

$$\frac{q(z)}{q'(z)} = \frac{1 - 1/x^2}{p'(x)/p(x) - nk/(2x)}, \quad p'(x)/p(x) = \text{tr}(P(x)^{-1}P'(x)). \quad (10)$$

This equation enables one to apply the EAI to the polynomial $q(z)$. Once its roots $z_1, \dots, z_{nk/2}$ have been computed, the eigenvalues of $P(x)$ are given by the pairs $(x_i, 1/x_i)$ which are the roots of the quadratic polynomial $x^2 - z_i x + 1$. This approach has the advantage to work with an approximation vector of half the size and to deliver the solution as pairs $(x, 1/x)$.

It is important to point out that the applications $z \rightarrow x = (z \pm \sqrt{z^2 - 4})/2$ is ill conditioned at $z = \pm 2$. Therefore, loss of accuracy is expected near $x = \pm 1$. In this case, a refinement step is advisable. Such a refinement may be implemented by an unstructured version of the EAI, by the naive structured EAI, or with other structured refinement methods [14].

If nk is odd, then -1 is necessarily an eigenvalue of the palindromic PEP and there is no need to approximate it. To calculate approximations of the remaining $nk - 1$ eigenvalues, there are two possible strategies.

As a first possibility, one may consider the new matrix polynomial $Q(x) = (x+1)P(x)$ which has even degree. The eigenvalues of $Q(x)$ are those of $P(x)$ and the eigenvalue -1 with multiplicity increased by n . Therefore, the previously described technique can be applied. Only $nk - 1$ roots of $\det(Q(x))$ are needed, because $n + 1$ roots are a priori known to be equal to -1 . Thus, one could apply the EAI (2) or (3) with an approximation vector y of $n(k + 1)$ components of which $n + 1$ are set equal to -1 in order to immediately achieve implicit deflation of the roots; or, working in the variable z in order to extract the structure, the SEAI (10) can be used setting $(n + 1)/2$ starting points equal to -2 .

A second possibility is to set $x := w^2$ and to consider the eigenvalues of the polynomial $Q(w) = P(x(w))$. The scalar polynomial $q(w) := \det Q(w)$ has

$2nk$ roots, which are the square roots of the solutions of the original equation $p(x) = 0$ that we have to solve. In particular only $2nk - 2$ roots are to be determined, since $q(w) = 0$ has two known solution at $w = \pm i$. It is useful to set $z := (w + 1/w)^2 = x + 1/x + 2$. Defining

$$\tilde{q}(w) := \frac{q(w)}{w^{nk+1} + w^{nk-1}},$$

it is easy to check that $r(z) := \tilde{q}(w(z))$ is a polynomial in z . Therefore we may restrict the attention to computing the roots of $r(z)$. Once they have been computed, the evaluation of the function $w(z)$ at these roots provides the roots of $q(w)$. The evaluation of $x(w)$ at these latter roots yields the sought eigenvalues of $P(x)$. In order to compute the roots of $r(z)$ we may apply the EAI to the polynomial $r(z)$. The following equations provides a tool to compute the Newton correction $r(z)/r'(z)$ needed by the EAI.

$$\frac{r(z)}{r'(z)} = \frac{2w(1 - 1/w^4)}{q'(w)/q(w) - [(nk + 1)w^2 + nk - 1]/(w^3 + w)},$$

or in terms of the original variable x

$$\frac{r(z)}{r'(z)} = \frac{1 - 1/x^2}{p'(x)/p(x) - [(nk + 1)x + nk - 1]/(2x^2 + 2x)}.$$

At the moment we have no clear elements to say which of the two possibilities is more convenient. We plan to investigate in this direction.

We conclude this subsection mentioning that also *antipalindromic* and *anti-T-palindromic* polynomials ($\text{Rev}P(x) = -P(x)$ and $\text{Rev}P(x) = -P(x)^T$) have a $\{x, 1/x\}$ symmetry. Their determinants are pure palindromic if n is even and antipalindromic if n is odd [26]. The former case is exactly the same as above. The latter case is also easy, because a scalar antipalindromic polynomial is always equal to $x - 1$ times a scalar pure palindromic polynomial. Moreover, it is possible to prove [26] that 1 is always a root of a scalar antipalindromic polynomial, and -1 is always a root of even-grade antipalindromic polynomial, so according to the grade there are either one or two exceptional eigenvalues with odd multiplicity. Therefore, it is easy to extend our technique to this class.

3.3. Hamiltonian/skew-Hamiltonian and even/odd

An even (odd) polynomial is such that P_j is symmetric for all even (odd) values of j and is skew-symmetric for all odd (even) j . Similarly, the coefficients of a Hamiltonian/skew-Hamiltonian polynomial are, alternatively, Hamiltonian and skew-Hamiltonian matrices. The classes of even-dimensional even/odd polynomials are easily mapped onto the classes of Hamiltonian/skew Hamiltonian polynomials by a multiplication by J . Amongst the huge literature on these classes of polynomials see, for instance, [31, 32, 33, 43] and the references therein. Classical eigenvalue problems for Hamiltonian matrices [48] are a special case of skew-Hamiltonian PEPs.

The matrix polynomials belonging to these classes have eigenvalues coming in pairs $(x, -x)$. In particular, if nk is odd, then either $x = 0$ (if P_0 is skew-symmetric) or $x = \infty$ (if P_k is skew-symmetric) is an eigenvalue. Notice that this is never the case for Hamiltonian/skew-Hamiltonian polynomials, because they are only defined for even n .

Let $z := x^2$. Just like the T-palindromic case, also for even/odd polynomials it is possible to follow the ideas exposed in [13] and build a new matrix polynomial $M(z)$ whose determinant is equal to $p(x(z)) \cdot p(x(z))$. The following result demonstrates the way it can be done for an even polynomial.

Proposition 2. *Let $P(x)$ be an even matrix polynomial, and let $z = x^2$. Define $B(z) := \frac{P(x(z)) + P^T(x(z))}{2}$ and $C(z) := \frac{1}{x(z)} \frac{P(x(z)) - P^T(x(z))}{2}$, so that $P(x) = B(x^2) + xC(x^2)$. Then $M(z) := \begin{pmatrix} B(z) & zC(z) \\ C(z) & B(z) \end{pmatrix}$ is a skew-Hamiltonian matrix polynomial such that $\det M(z) = [p(x(z))]^2$. If $0 \neq x_0$ in \mathbb{C} is an eigenvalue for $P(x)$ associated with a canonical set of Jordan chains of length ℓ_1, \dots, ℓ_k then x_0^2 is an eigenvalue for $M(z)$ and its Jordan structure is the union of the Jordan structures of $P(x)$ at x_0 and at $-x_0$.*

Moreover:

1. concerning eigenvectors associated with any finite nonzero eigenvalue x_0 , $P(x_0)v_0 = 0$ and $P(-x_0)w_0 = 0$ if and only if $v_1 = (x_0 v_0^T, v_0^T)^T$ and $w_1 = (-x_0 w_0^T, w_0^T)^T$ are two linearly independent eigenvectors such that $M(x_0^2)v_1 = 0 = M(x_0^2)w_1$;
2. $M(z)$ is a $2n \times 2n$ matrix polynomial of degree $\deg M = [(k+1)/2]$;
3. writing $M(z) = \sum_{j=0}^{\deg M} M_j z^j$, the relation $M_j = \begin{pmatrix} P_{2j} & P_{2j-1} \\ P_{2j+1} & P_{2j} \end{pmatrix}$ holds for all $0 \leq j \leq \deg M$, where $P_j = 0$ if $j < 0$ or $j > k$;
4. if k is odd, $M(z)$ has at least n (respectively, $n+1$) eigenvalues at infinity if n is even (respectively, odd).

The proof can be obtained by adapting the arguments used in [13] for T-palindromic polynomials to the even case; we skip the details here.

Similar results can of course be obtained for odd and Hamiltonian/skew-Hamiltonian matrix polynomials. Applying the EAI to $M(z)$ allows us to extract the structure. An alternative approach, that avoids possible issues about loss of accuracy for very large eigenvalues (this time $M(z)$ has extra infinite eigenvalues only if k is odd), is once again the implicit use of the squaring transformation. Namely, if nk is even (which is always satisfied for Hamiltonian/skew-Hamiltonian polynomials), then defining $z := x^2$ one finds that $q(z) := p(x(z))$ is a polynomial for $x(z) = \sqrt{z}$ or $x(z) = -\sqrt{z}$. Thus, $p'(x)/(2xp(x)) = q'(z)/q(z)$, so that the Newton correction for the polynomial $q(z)$ is readily available

$$q(z)/q'(z) = 2xp(x)/p'(x) = 2x/\text{tr}(P(x)^{-1}P'(x))$$

and the Ehrlich-Aberth algorithm can be implicitly applied to the polynomial $q(z)$ in order to compute its roots $z_1, \dots, z_{nk/2}$. This way, the roots of $p(x)$ are readily available in pairs as $(\sqrt{z_i}, -\sqrt{z_i})$.

In other situations, one eigenvalue is necessarily either 0 (if $P(x)$ is odd and nk is odd) or ∞ (if $P(x)$ is even and nk is odd); thus, there is no need to approximate it. It may also happen that there is one uncoupled eigenvalue at 0 and another one at ∞ (e.g. if $P(x)$ is odd, n is odd and k is even). In the case of an extra eigenvalue at 0, to approximate the other eigenvalues one can notice that $q(z) := p(\sqrt{z})/\sqrt{z}$ is a polynomial and that $q'(z)/q(z) = (1/2x^2)(xp'(x)/p(x) - 1)$. This yields the Newton correction for $q(z)$ as

$$q(z)/q'(z) = 2x/(p'(x)/p(x) - 1/x) = 2x/(\text{tr}(P(x)^{-1}P'(x)) - 1/x), \quad z = x^2,$$

which enables one to apply the EAI to $q(z)$ by using an approximation vector of length $(nk - 1)/2$. As in the palindromic case, there is also the alternative option to consider the polynomial $xP(x)$ which is even (odd) if $P(x)$ is odd (even). The new polynomial $xP(x)$ has n additional eigenvalues at 0 that are known and can therefore be immediately deflated.

3.4. Unified approach to any structure

More in general, let $\mathbb{C}^* := \mathbb{C} \cup \{\infty\}$ and let $f : \mathbb{C}^* \rightarrow \mathbb{C}^*$ be any self-inverse function, that is $f(f(x)) = x \quad \forall x \in \mathbb{C}^*$. An example is the subclass of rational functions $f(x) = \frac{ax+b}{cx-a}$, which are self-inverse whenever $a^2 + bc \neq 0$. If we additionally require f to be analytic, having such a form is not only a sufficient condition, but it is also necessary (unless $f(x) = x$) for f to be self-inverse. This follows from the fact that Möbius functions (i.e., rational functions of degree 1) are the only automorphisms of \mathbb{C}^* .

Suppose that, because of some structure in the coefficients of $P(x)$, all eigenvalues come in pairs $\{\lambda, f(\lambda)\}$. Eigenvalues such that $\lambda = f(\lambda)$ are called *exceptional*, and are allowed to appear with any multiplicity.

Such a possibility justifies the requirements that

1. $f(x)$ is analytic, so that either it is the identity function or it has a finite number of fixed points, and
2. there is a way to identify which exceptional eigenvalues, if any, appear with odd multiplicity.

In fact, exceptional eigenvalues can otherwise become a problem. For instance, consider a real matrix polynomial associated to the non-analytic function $f(x) = x^*$: the method meets problems in this case. The reason is that, since all the real line is exceptional, there is no way to state *a priori* which exceptional (i.e. real) eigenvalues, if any, appear without being part of a complex conjugate eigencouple.

If f is analytic, the implicit change of variable method that we have described for the special cases $f(x) = -x$ and $f(x) = 1/x$ can be generalised in the following way. Suppose first that $a \neq 0$, and define $z(x) := \frac{ax^2+bx}{cx-a} = xf(x)$. Let $x(z)$ denote any of the two branches of the inverse function of $z(x)$. Then if there are no eigenvalues with odd multiplicity one can see that $q(z) := \frac{p(x(z))}{(cx(z)-a)^{nk/2}}$ is a polynomial; therefore, the EAI can be applied to $q(z)$ and the eigenvalues can be found inverting the rational function $z(x)$. Otherwise (e.g. if nk is odd) there

must be some exceptional eigenvalues that can be treated with techniques akin to those described for the special cases previously considered. If on the contrary $a = 0$, let $z(x) := \frac{cx^2+b}{cx} = x + f(x)$. Once again if all the eigenvalues come out in couples then $q(z) := \frac{p(x(z))}{x(z)^{nk/2}}$ is a polynomial; otherwise one can simply deal with the exceptional eigenvalues with known odd multiplicity by using techniques analogous to those described in the previous subsections. Notice that the fixed points of $f(x)$ may lead to computational problems, since they are double roots in the equation $z(x) = \zeta$. Refinements of some kind are advisable there. See also [14].

The explicit method may also be extended to the general case. This is the subject of a future research project.

4. Numerical experiments

We have performed extensive numerical experiments in order to check the efficiency and the accuracy of our implementation of the EAI. Further tests have been performed to confirm the ability of our method to exploit structures in the coefficients and to respect structures in the spectrum when approximating it.

4.1. Efficiency

The complexity of the proposed algorithm is of order $tn^3 + tkn^2$, where t is the total number of times that a trace computation is needed before (vectorial) convergence. There is empirical evidence that t heavily depends on the choice of the initial approximation. For the case of scalar polynomials, the use of suitable strategies [3, 4] leads to a linear dependence of t with respect to the total number of roots.

Experiments we made with our implementation, with starting points determined by the Newton polygonal, suggest that this is also the case of the EAI applied to a matrix polynomial. This means that the computational complexity of the EAI is $\mathcal{O}(kn^4 + k^2n^3)$, leading to great computational advantages for $k \gg \sqrt{n}$. As noticed in 2.6, if on the contrary $k \lesssim \sqrt{n}$ other more focused implementation of the EAI are possible, with cubic efficiency in kn .

In order to confirm such predictions, we have compared our implementation of the EAI and Matlab¹'s QZ implementation *polyeig* on random matrix polynomials of high degree and small dimension. The experiments have been performed on a machine with CPU Intel Xeon 2.80GHz and system Linux Debian 6.02. For very large values of k , we did not actually run *polyeig* due to the very large forecast computation times, but we extrapolated the times from the other experiments; in fact, when doubling the value of k we can expect that the running time of the QZ algorithm grows approximately with a factor 8. Such extrapolated values are marked with a * in the following tables.

¹Matlab is a registered trademark of The MathWorks, Inc.

Computation times for $n = 2$			Computation times for $n = 5$		
k	EAI	polyeig	k	EAI	polyeig
50	0.018 s	0.015 s	20	0.062 s	0.010 s
100	0.044 s	0.064 s	40	0.121 s	0.057 s
200	0.111 s	0.369 s	80	0.312 s	0.370 s
400	0.360 s	4.35 s	160	0.920 s	4.39 s
800	1.29 s	51.9 s	320	2.92 s	44.0 s
1600	4.76 s	437 s	640	10.3 s	398 s
3200	18.4 s	$\mathcal{O}(50 \text{ min})^*$	1280	38.1 s	$\mathcal{O}(50 \text{ min})^*$
			2560	148 s	$\mathcal{O}(7 \text{ hours})^*$
			5120	575 s	$\mathcal{O}(2 \text{ days})^*$

The values in the tables above are in agreement with our prediction that the computation time should asymptotically grow as k^2 . Moreover, the experimentation also confirms that, for a given value of n , the ratio of the time needed by EAI with respect to the time needed by the QZ algorithm exhibits an asymptotic growth that is approximately linear in k . This effect is taken to the extreme in the case $n = 5$, $k = 5120$. Had we used Matlab's *polyeig*, it would have taken several days of computation time on our machine to solve such a problem. Our implementation of the EAI gave the approximated eigenvalues in less than 10 minutes.

4.2. Accuracy

In order to test the accuracy of our implementation we used the Matlab toolbox NLEVP[2]. This toolbox has been recently proposed by its authors as an interesting set of benchmark problems that may be used as a standard test for new methods for nonlinear eigenvalue problems. It contains data coming from practical applications as well as model problems known to have peculiar properties.

Amongst the many nonlinear eigenproblems contained in NLEVP, we have selected all the square polynomial eigenproblems with $n < 25k^2$. We discarded PEPs with a larger ratio n/k^2 because they could be better dealt with by a different implementation of EAI, via a preliminary linearization. The test suite selected with this criterion consists of 29 problems plus the 2 problems butterfly and wiresaw1 that, being structured, will be treated in the next subsection.

In all the parameter-dependent problems in the library the default values of the parameters were selected. All methods were directly applied to the original matrices as saved in the library, without preprocessing them with any scaling. Forward errors are evaluated by comparing the approximations with either theoretically known values, when available, or values computed in variable precision arithmetic (VPA) with Matlab's symbolic toolbox.

The graphs below are in logarithmic scale. Whenever the absolute error for a certain eigenvalue λ appeared to be numerically zero, i.e. it was less than λ times the machine epsilon $\epsilon = 2^{-52} \simeq 2.22\text{e} - 16$, we formally set it equal to $\frac{\lambda\epsilon}{2}$. Only absolute errors for the finite eigenvalues are shown in the figures.

For the 3 problems with $k \geq 3$, the eigenvalue forward errors were computed for both the EAI and the QZ method (as implemented in *polyeig*). Absolute errors for our implementation of the EAI are marked with a red * symbol, while absolute errors for *polyeig* are marked with a blue + symbol. For this set of experiments, we picked starting points on the unit circle. In our experience the order of magnitude of the forward error is not significantly affected by the choice of the starting points, even though for some problems other choices led to slight improvements (not discussed here).

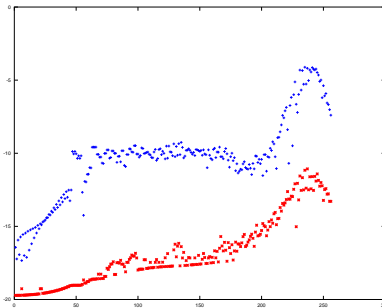


Fig. 1. *Forward absolute errors for the problem orr sommerfeld*

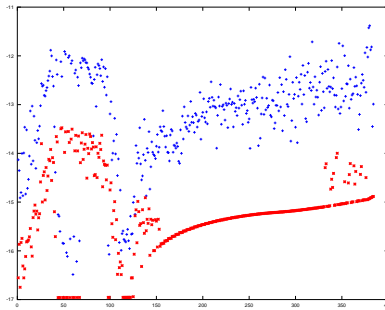


Fig. 2. *Forward absolute errors for the problem plasma drift*

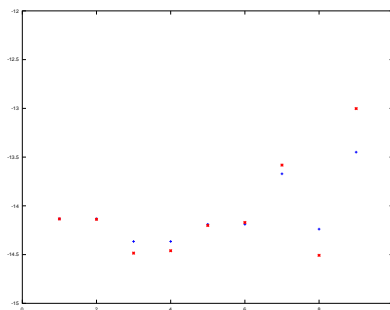


Fig. 3. *Forward absolute errors for the problem relative pose 5pt*

For the 26 problems with $k = 2$, three methods were compared by computing their forward errors with the same method as above: *polyeig* (blue + symbol), EAI (red * symbol) and the software *quadeig* by Hammarling, Munro and Tisseur [16], specifically designed for quadratic PEPs (black x symbol). Although we did not alter the coefficients given as input to any method, for most problems *quadeig* have performed scaling by the default settings of its internal algorithm, that prescribe scaling under certain conditions; see [16].

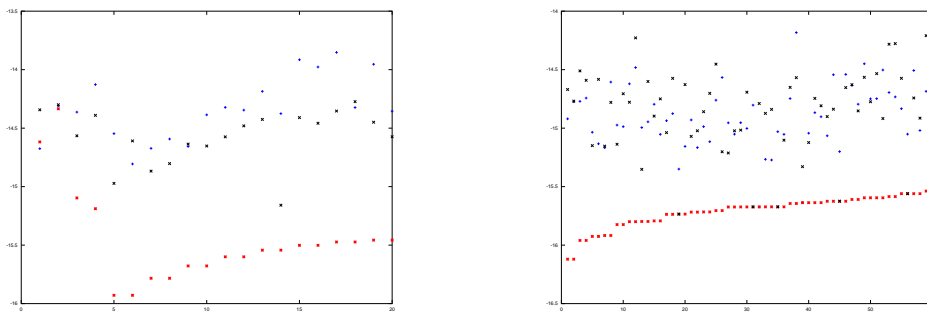


Fig. 4. *Forward absolute errors for the problems acoustic wave 1d (left) and acoustic wave 2d (right)*

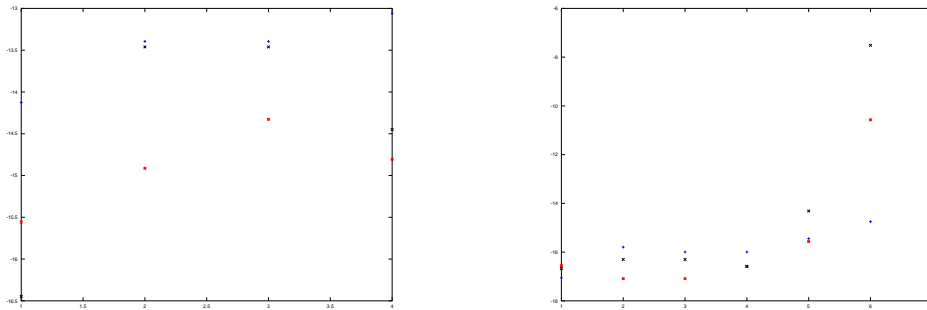


Fig. 5. *Forward absolute errors for the problems bicycle (left) and bilby (right)*

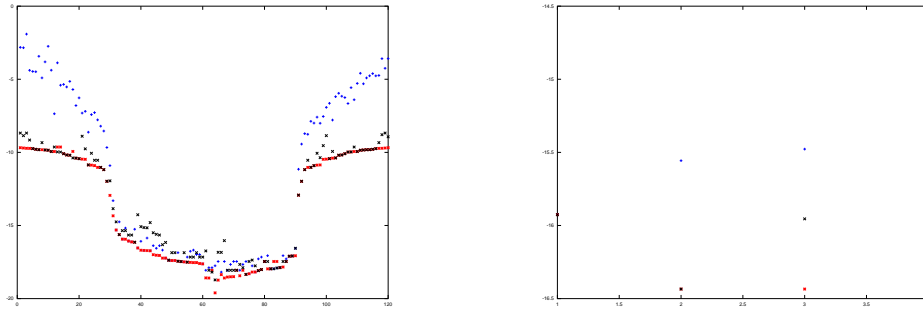


Fig. 6. Forward absolute errors for the problems *cd player* (left) and *closed loop* (right)

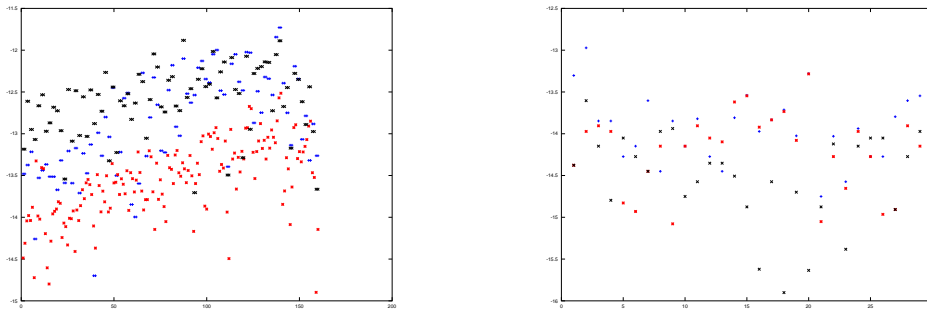


Fig. 7. Forward absolute errors for the problems *dirac* (left) and *gen hyper 2* (right)

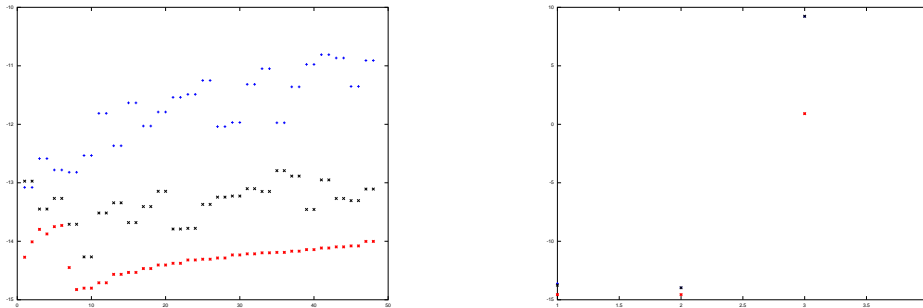


Fig. 8. Forward absolute errors for the problems *hospital* (left) and *intersection* (right)

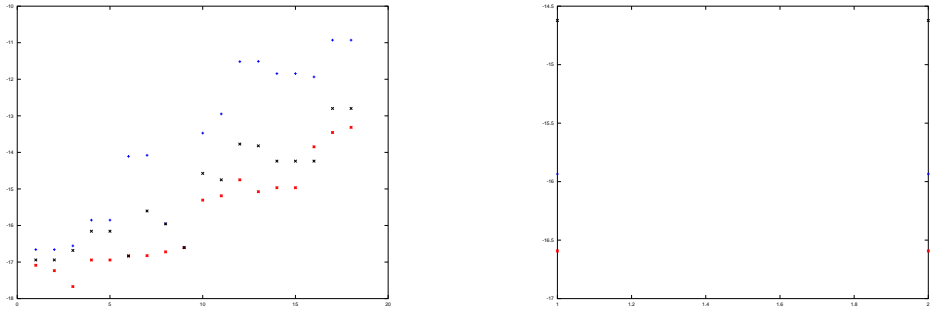


Fig. 9. Forward absolute errors for the problems metal strip (left) and mobile manipulator (right)

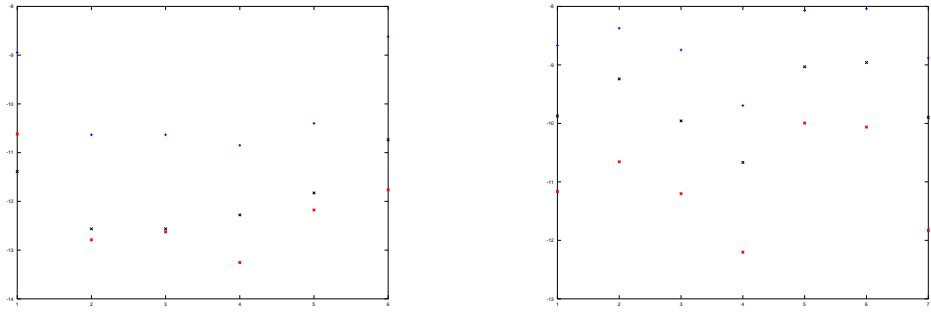


Fig. 10. Forward absolute errors for the problems omnicam1 (left) and omnicam2 (right)

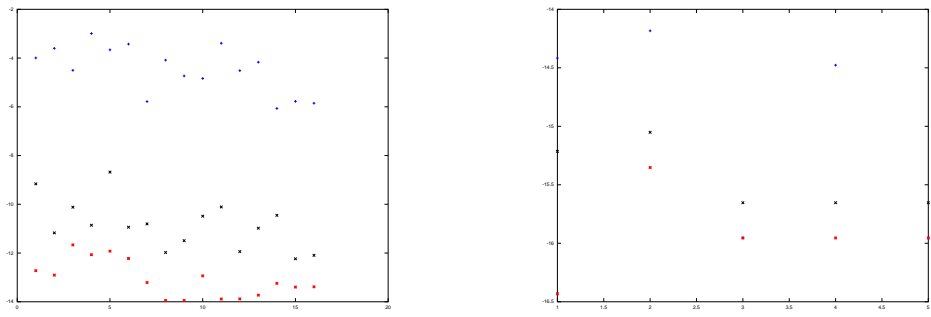


Fig. 11. Forward absolute errors for the problems power plant (left) and qep1 (right)

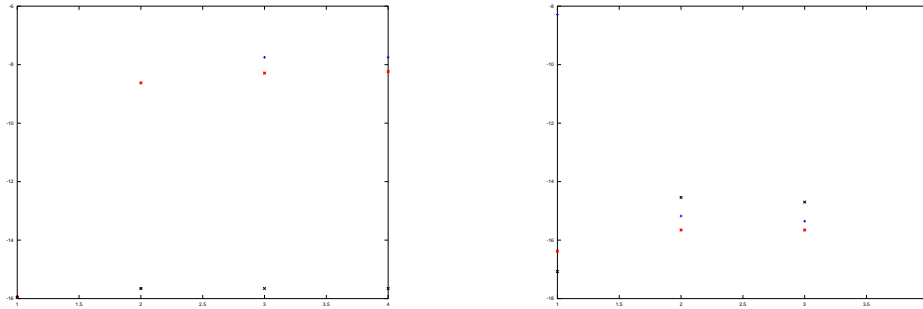


Fig. 12. Forward absolute errors for the problems *qep2* (left) and *qep3* (right)

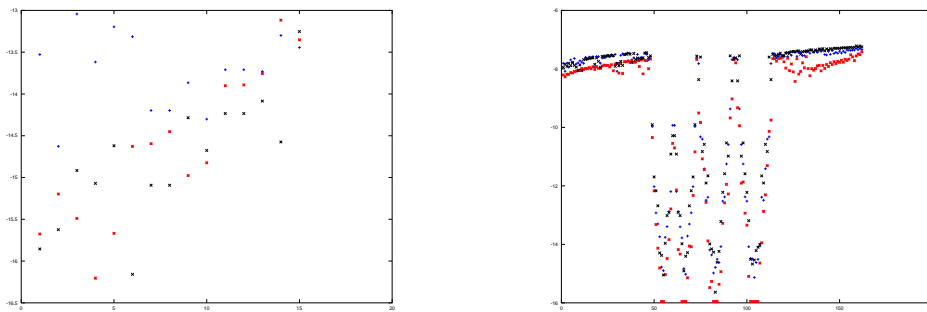


Fig. 13. Forward absolute errors for the problems *relative pose 6pt* (left) and *sign1* (right)

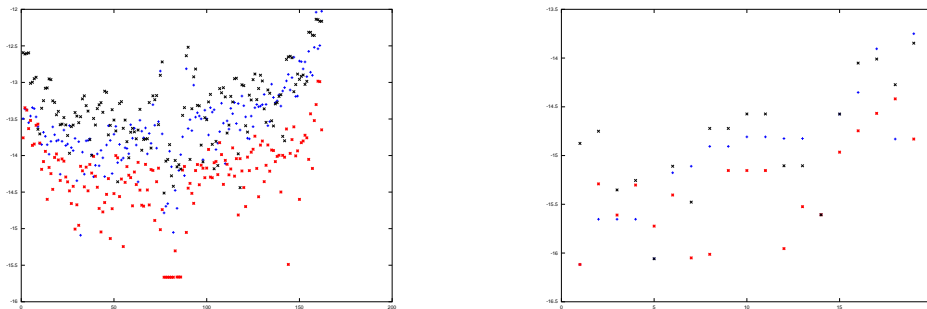


Fig. 14. Forward absolute errors for the problems *sign2* (left) and *sleeper* (right)

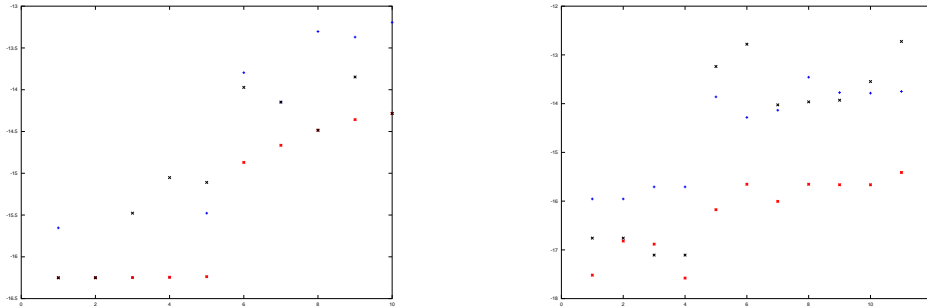


Fig. 15. Forward absolute errors for the problems *spring* (left) and *spring dashpot* (right)

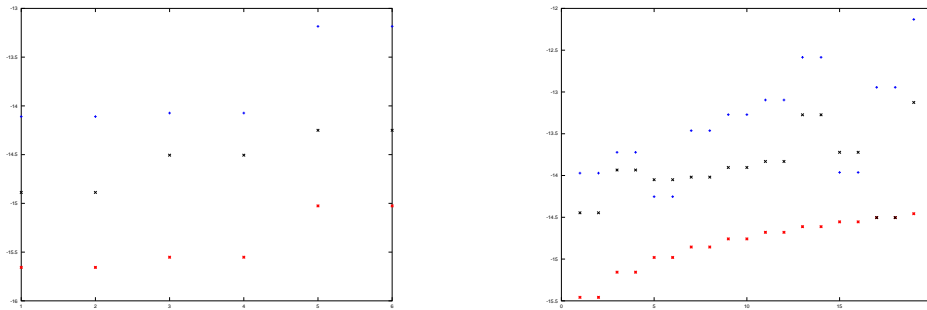


Fig. 16. Forward absolute errors for the problems *wing* (left) and *wiresaw2* (right)

As can be seen by the figures above, the approximations of the EAI are competitive, and often more accurate than the approximations of the QZ. In some cases, the improvement is remarkable. We report in the following table the maximal relative error and the average relative error for all the finite (i.e. neither numerically zero nor numerically infinite) eigenvalues of the 29 considered problems, and for both the EAI and the QZ. The average relative error is defined as the geometric mean of all relative errors; numerically zero relative errors have been counted as relative errors equal to $\epsilon/2$. The values reported for the QZ for quadratic problems correspond to the algorithm, picked between *polyeig* and *quadeig*, that achieved the best performance in terms of average relative error for the given problem. As can be deduced by the above pictures and coherently with the results on backward errors presented in [16], such best performance was achieved generally, but not always, by the latter algorithm.

Problem	Rel. errors, EAI		Rel. errors, QZ	
	Max.	Avg.	Max.	Avg.
acoustic wave 1d	1.0e-14	2.1e-16	1.1e-14	1.7e-15
acoustic wave 2d	$\epsilon/2$	$\epsilon/2$	3.2e-15	7.4e-16
bicycle	1.0e-15	4.0e-16	7.6e-15	1.1e-15
bilby	2.4e-14	3.5e-16	5.1e-15	1.8e-15
cd player	5.3e-16	1.2e-16	4.0e-14	3.3e-16
closed loop	$\epsilon/2$	$\epsilon/2$	3.4e-16	1.5e-16
dirac	4.1e-14	5.9e-15	1.9e-13	2.9e-14
gen hyper 2	2.5e-14	9.3e-16	2.4e-15	4.9e-16
hospital	2.7e-15	1.6e-16	2.0e-14	1.4e-15
intersection	4.8 e-9	4.5 e-13	1.0	2.4e-8
metal strip	6.3e-16	1.7e-16	2.3e-15	6.7e-16
mobile manipulator	$\epsilon/2$	$\epsilon/2$	5.1e-16	5.1e-16
omnicam1	9.1e-11	1.2e-12	4.3e-9	6.4e-13
omnicam2	3.9e-10	2.3e-15	4.0e-9	1.2e-13
orr sommerfeld	5.0e-12	9.1e-16	4.8e-5	2.8e-9
plasma drift	3.4e-13	5.1e-16	1.3e-11	4.7e-14
power plant	8.3e-14	1.1e-15	6.1e-11	1.9e-13
qep1	8.9e-16	1.7e-16	1.8e-15	5.1e-16
qep2	5.8e-9	5.3e-11	2.2e-16	1.9e-16
qep3	3.9e-9	1.0e-14	8.0e-10	3.2e-14
relative pose 5pt	2.9e-14	6.8e-15	1.6e-14	6.3e-15
relative pose 6pt	7.5e-14	1.9e-14	1.2e-13	1.4e-14
sign1	3.8e-8	1.1e-10	5.0e-8	3.9e-10
sign2	4.5e-14	2.8e-15	3.1e-13	1.3e-14
sleeper	8.0e-16	2.8e-16	1.8e-15	5.6e-16
spring	$\epsilon/2$	$\epsilon/2$	1.7e-15	3.5e-16
spring dashpot	5.6e-15	3.1e-16	2.8e-13	1.4e-14
wing	$\epsilon/2$	$\epsilon/2$	1.2e-15	8.1e-16
wiresaw2	$\epsilon/2$	$\epsilon/2$	2.4e-15	9.2e-16

As the results above show, the EAI was generally able to improve the accuracy of the approximations with respect to the QZ method. The only problems where the EAI achieved an average performance worse than the QZ are QEP2 (loss accuracy on the multiple eigenvalue 1 with respect to *quadeig*; *polyeig* has problems as well) and gen hyper 2.

The problems in NLEVP do not have high degree, so the condition $k^2 \gg n$ is not met. Therefore, in contrast with the high degree case, for those problems using the EAI as a primary algorithm does not bring advantages in term of computation time; on the contrary the implementation discussed in this paper is slower than QZ if $n \gg k^2$. Also, in these cases a linearization-based version of the EAI would be more efficient than the polynomial-based implementation discussed in the present paper. However, it is worth noticing that the numerical experiments showed that very often it improved the accuracy achieved by *polyeig*

and/or *quadeig*; in many cases, the EAI is able to compute correctly all the digits of the eigenvalue up to machine precision. This suggests that, when $k^2 \lesssim n$, it is possible to use the EAI as a refinement algorithm in order to improve the approximations obtained by the QZ. Using such values as starting points offers of course a very good choice, lowering the number of overall scalar iteration needed before convergence and therefore improving the efficiency of the EAI.

4.3. Structured case

Discussions and analyses of the behaviour of different structured versions of the EAI applied to structured PEPs have already appeared in [13] for palindromic polynomials and in [36] for even/odd polynomials.

Here we further verify the reliability of our method with two tests. The first test relies on the NLEVP problems butterfly and wiresaw1, which are even [2]. The next figures compare four algorithms applied to these problems: *polyeig*'s QZ (blue + symbol), unstructured EAI (UEAI, red * symbol), the structured matrix method URV applied to an even linearization [43] (black x symbol), and a structured version of the EAI relying on the change of variable $z = x^2$ method (SEAI, green o symbol).

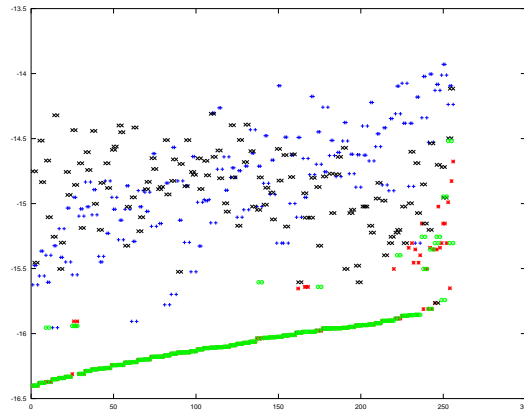


Fig. 17. Forward absolute errors for the problem butterfly

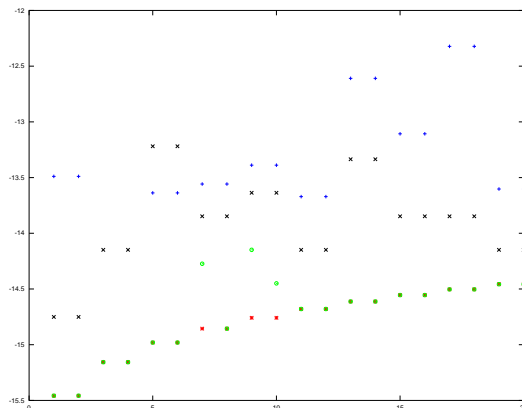


Fig. 18. *Forward absolute errors for the problem wiresaw1*

It is clear from the figures above that for the problem butterfly the EAI was more accurate than the two matrix methods, but structured methods did not improve much the accuracy of each unstructured counterpart. This suggests that for these matrix polynomials the unstructured condition numbers for the eigenvalues are not much different from the structured condition numbers; therefore, structured methods do not improve much the accuracy, even though they improve the efficiency.

In the second test, a matrix polynomial $W(x)$ with $n = 2$ and $k = 10$ was built in such a way that its eigenvalues appear in couples of the form $\{\lambda, \frac{\lambda+1}{\lambda-1}\}$. In order to devise a problem not too easy to solve numerically, the determinant of the polynomial was designed to be Wilkinson-like: $\det(W(x)) = const. \cdot \theta(x) \cdot \theta(\frac{x+1}{x-1})$, $\theta(x) = x \cdot \prod_{j=2}^{10} (x - j)$. The next figure shows the absolute errors of the computed approximations with respect to the known exact eigenvalues for three methods: QZ (polyeig, blue + symbol), UEAI (red * symbol) and SEAI relying on the change of variable $z = \frac{x^2+x}{x-1}$ (green o symbol). Numerically zero errors were formally set equal to $\epsilon/2$.

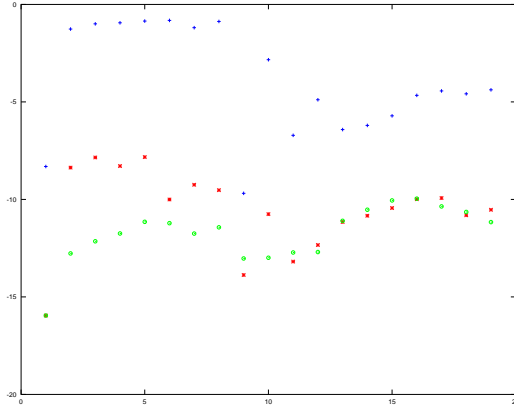


Fig. 19. *Forward absolute errors for the structured problem $\det W(x) = 0$*

The following table reports the relative errors of the three methods considered above for all the eigenvalues but 0 (all the three algorithms detected the zero eigenvalue with an absolute error smaller than the machine epsilon). It also reports the relative error for the Matlab's function *eig* (without scaling) applied to a suitable linearization, chosen according to the prescriptions of [18]. Notice that here all the nonzero eigenvalues have modulus ≥ 1 , so the suggested (near-to-optimal) linearization in the space \mathbb{DL} according to [18] would in principle be the pencil in \mathbb{DL} corresponding to the ansatz vector e_1 (see [28] for further details). Unluckily, since one eigenvalue is zero, that pencil is not a linearization at all [28]. Following the suggestions on conditioning of [18] and the theory on linearizations of [28], we have therefore taken the slightly perturbed vector $e_1 + 2^{-23}e_{10}$ as an ansatz vector. The factor 2^{-23} has been heuristically chosen picking the integer $\alpha \leq 52$ that minimises the average relative error when applying *eig* to the linearizations in \mathbb{DL} associated to the ansatz vectors $e_1 + 2^{-\alpha}e_{10}$.

Eigenvalue	R. e., polyeig	R. e., eig	R. e., UEAI	R. e., SEAI
-1	4.9e-9	$\epsilon/2$	$\epsilon/2$	$\epsilon/2$
11/9	4.5e-2	4.8e-8	3.5e-9	1.4e-13
5/4	8.1e-2	2.2e-7	1.1e-8	5.7e-13
9/7	8.9e-2	4.2e-7	4.0e-9	1.4e-12
4/3	1.1e-3	4.1e-7	1.1e-8	4.6e-12
7/5	9.9e-2	2.2e-7	7.1e-11	5.1e-12
3/2	8.9e-2	6.3 e-8	3.8e-10	2.5e-12
5/3	3.9e-2	8.9e-9	1.8e-10	1.1e-12
2	1.0e-10	4.9e-10	8.8e-12	5.1e-14
2	7.3e-4	2.3e-12	6.7e-15	4.7e-14
3	4.3e-6	6.3e-12	2.2e-14	6.3e-14
3	6.5e-8	5.8e-12	1.5e-13	6.7e-14
4	9.6e-8	1.5e-10	1.7e-12	2.0e-12
5	1.3e-7	8.1e-10	2.9e-12	5.9e-12
6	3.2e-7	2.2e-9	6.1e-12	1.5e-11
7	3.1e-6	3.3e-9	1.5 e-11	1.6e-11
8	4.6e-6	2.7e-9	1.5e-11	5.5e-12
9	2.9e-6	1.1e-9	1.7e-12	2.5e-12
10	4.2e-6	1.7e-10	3.0e-12	6.9e-13
Average	3.8e-5	9.6e-10	5.5e-12	4.1e-13

We may conclude that on this structured problem the EAI outperforms the QZ method for what concerns accuracy. Apparently *polyeig* struggles quite a bit here (which is coherent with the results of [18]), while the strategy of [18] works better, but still worse than the EAI. Although there are some approximations that do not benefit from the use of the structured version of the Ehrlich-Aberth algorithm, the SEAI has an overall advantage in accuracy over the UEAI, besides the obvious efficiency advantage. The computation time for the SEAI was about one third of the computation time for the UEAI.

5. Conclusions and lines of future research

We have proposed and tested a generalisation of the Ehrlich-Aberth method to polynomial eigenvalue problems. Both theoretical arguments and numerical experiments show that the Ehrlich-Aberth algorithm is more efficient than customary method for high-degree PEPs, since its complexity is only quadratic in k . Numerical experiments also suggest that in many situations the new method provides more accurate approximations, and therefore it may also be used as a refinement method for low-degree PEPs.

On the other hand, problems arise in the treatment of multiple eigenvalues. Current research is focused on this issue. Currently, our algorithm exploits a heuristic device to deal with multiple eigenvalues at 0 or ∞ , which is the most common occurrence in practice. For the NLEVP problems, very good results were obtained also when multiple eigenvalues were present.

Another line of current research regards the possible use of other root-finding algorithms. For instance, we can cite the modified Ehrlich-Aberth iteration [29, 38], the Durand-Kerner iteration [29, 40] or simultaneous root finders based on higher order methods in the Householder family, e.g. the Halley method [40]. It is advocated [29, 40] that some of the above mentioned method have order of convergence higher than 3. However, in practice we did not see a significant improvement on the total number of scalar iterations with respect to the EAI; sometimes, performances were definitely worse than the EAI. Further work is needed to compare the various possibilities in special cases like structured matrix polynomials.

Important issues on which we plan to keep on working are improving the design of reliable stop conditions, a posteriori error bounds and choices of starting approximations. About the latter issue, in [6] some results of [44] on scalar polynomials will be generalised to matrix polynomials.

Finally, we have proposed and tested a structured version of the algorithm that is able to catch certain structures in the spectrum. New numerical experiments on the matter confirm and extend the results of [13, 36] about the efficiency and the accuracy of the structured EAI.

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