References

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2. M. Holz, K. Steffens and E. Weitz *Introduction to Cardinal Arithmetic* Birkhäuser 1999: 1.6-7, 6.2, 7.2, 8.1, 9.1-2

The power in base two

The cardinal $2^\nu = |\mathcal{P}(\nu)|$ satisfies the monotonicity condition

$$(d1) \qquad \mu \leq \nu \implies 2^\mu \leq 2^\nu$$

together with the inequalities

$$(d2) \qquad \aleph_0^\nu = 2^\nu \geq \text{cof}\, 2^\nu > \nu$$

The behaviour of the function $\nu \mapsto 2^\nu$ on regular cardinal is completely free apart of the above constraints, namely

Theorem. *(Easton) Let $F : Reg \to Card$ be a (class) function satisfying (d1) and (d2). Then it is consistent with ZFC that $2^\nu = F(\nu)$ for all regular cardinals ν.* □

The singular cardinal case

Lemma 1. *Put* $\gimel(\xi) = \xi^{\mathrm{cof}\,\xi}$ *and* $2^{<\nu} = \sup\{2^\xi \mid \xi < \nu\}$. *Then*

(d3) $$2^\nu = (2^{<\nu})^{\mathrm{cof}\,\nu}$$ *for all cardinals* ν.

Proof. Let $\nu = \sum_{\alpha<\kappa} \nu_\alpha$, with $\nu_\alpha < \nu$ for $\alpha < \kappa \le \nu$. Then

$$2^\nu = 2^{\sum_{\alpha<\kappa}\nu_\alpha} = \prod_{\alpha<\kappa} 2^{\nu_\alpha} \le (2^{<\nu})^\kappa \le 2^{\nu\cdot\kappa} = 2^\nu$$ □

Theorem 1. *(Buchowski-Hechler) Let* ν *be singular; then*

(d4) $$2^\nu = \begin{cases} 2^{<\nu} & \text{if } \exists\kappa < \nu\ \forall\xi\ (\kappa \le \xi < \nu \Rightarrow 2^\xi = 2^\kappa) \\ \gimel(2^{<\nu}) & \text{otherwise.} \end{cases}$$

Proof. Both cases follow from (d3): if 2^ξ is not eventually constant below ν, then $\mathrm{cof}\,\nu = \mathrm{cof}\,2^{<\nu}$, and $(2^{<\nu})^{\mathrm{cof}\,\nu} = \gimel(2^{<\nu})$; if it is not, then $(2^{<\nu})^{\mathrm{cof}\,\nu} = (2^\kappa)^{\mathrm{cof}\,\nu} = 2^{\kappa\cdot\mathrm{cof}\,\nu} = 2^{<\nu}$. □

So the power 2^ν for singular ν is determined by the function $\gimel$ on singular cardinals, together with the power $2^\kappa = \kappa^\kappa = \gimel(\kappa)$ of regular cardinals $\kappa < \nu$ (in fact $2^{<\nu} = \sup\{2^{\xi^+} \mid \xi < \nu\}$ for singular ν).

Cardinal power

The cardinal exponentiation κ^ν satisfies the obvious relations

$$(e1) \qquad \lambda \le \kappa \implies \lambda^\nu \le \kappa^\nu$$

$$(e2) \qquad \mu \le \nu, \implies \kappa^\mu \le \kappa^\nu$$

$$(e3) \qquad \xi < \kappa,\ \xi^\nu \ge \kappa \implies \xi^\nu = \kappa^\nu$$

together with the strict inequalities

$$(e4) \qquad \mathrm{cof}\,\kappa^\nu > \nu \quad and \quad \kappa^{\mathrm{cof}\,\kappa} > \kappa$$

It turns out that the *gimel* function $\gimel(\kappa) = \kappa^{\mathrm{cof}\,\kappa}$ completely determines the cardinal exponentiation.

(But clearly $\gimel(\kappa) = \kappa^\kappa = 2^\kappa$ for regular κ.)

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Preliminary lemmata

Lemma 2. *Assume that* $\nu < \operatorname{cof}\kappa$ *and let* $f : \nu \to \kappa$ *be given. Then there exists* $\alpha \in \kappa$ *s.t.* $f[\nu] \subseteq \alpha$, *whence* ${}^{\nu}\kappa \subseteq \bigcup_{\alpha\in\kappa} {}^{\nu}\alpha$. *Hence*

$$\nu < \operatorname{cof}\kappa \implies \kappa^{\nu} = \sum_{\xi<\kappa} \xi^{\nu}\xi^{+} \tag{e5}$$

Proof. The range of f cannot be cofinal in κ, so it is contained in some $\alpha \in \kappa$. Then (e5) follows, because $|{}^{\nu}\alpha| = |\alpha|^{\nu}$ and $\xi^{+} = |\{\alpha \mid |\alpha| = \xi\}|$. □

It follows the *Hausdorff formula* $(\kappa^{+})^{\nu} = \kappa^{\nu}\kappa^{+}$ and in general

$$(\aleph_{\alpha+n})^{\nu} = \aleph_{\alpha}^{\nu}\,\aleph_{\alpha+n} \quad \textit{for all } \alpha \textit{ and all } n \tag{e6}$$

Lemma 3. *Let* κ *be a limit cardinal, and let* $\nu \geq \operatorname{cof}\kappa$. *Then*

$$\nu \geq \operatorname{cof}\kappa \implies \kappa^{\nu} = (\sup_{\xi<\kappa} \xi^{\nu})^{\operatorname{cof}\kappa} \qquad (\kappa \textit{ limit}) \tag{e7}$$

Proof. Let $\kappa = \sum_{\gamma<\operatorname{cof}\kappa} \kappa_{\gamma}$, with $\kappa_{\gamma} < \kappa$ for all $\gamma < \operatorname{cof}\kappa$. Then

$$\kappa^{\nu} \leq (\prod_{\gamma<\operatorname{cof}\kappa} \kappa_{\gamma})^{\nu} = \prod_{\gamma<\operatorname{cof}\kappa} \kappa_{\gamma}^{\nu} \leq (\sup_{\xi<\kappa} \xi^{\nu})^{\operatorname{cof}\kappa} \leq \kappa^{\nu\operatorname{cof}\kappa} = \kappa^{\nu}.$$

□

Buchowski's theorem

Theorem 2. *(Buchowski)*

$$(e8)\qquad \kappa^\nu = \begin{cases} 2^\nu & \textit{if } \ \kappa \le 2^\nu \ \ (in\ particular\ if\ \nu \ge \kappa), \\ \kappa & \textit{if } \nu < \operatorname{cof}\kappa \ and\ \forall \xi < \kappa\, (\xi^\nu \le \kappa), \\ \gimel(\kappa) & \textit{if } \kappa > \nu \ge \operatorname{cof}\kappa \ and\ \forall \xi < \kappa\, (\xi^\nu < \kappa), \\ \gimel(\zeta) & \textit{otherwise, where } \zeta = \min\{\xi < \kappa \mid \xi^\nu \ge \kappa\}. \end{cases}$$

Proof.

(e1) gives the first case, (e5) the second case, (e7) the third case. If neither of them holds, then $\{\xi < \kappa \mid \xi^\nu \ge \kappa\} \ne \emptyset$, so (e1) gives $\kappa^\nu = \zeta^\nu$, with $\zeta > \aleph_0$ (otherwise the first case holds). Proceed by induction on κ, so the thesis holds for ζ: then $\xi^\nu < \zeta$ for all $\xi < \zeta$, by minimality of ζ. Moreover ζ is not in the first two cases, because $\zeta^\nu \ge \kappa > \zeta$, hence necessarily $\operatorname{cof}\zeta \le \nu < \zeta$, and $\zeta^\nu = \gimel(\zeta)$ by induction hypothesis. □

Remark that the last two cases may occur only when κ, resp. ζ are singular.

Special hypotheses

Assuming the **Generalized Continuum Hypothesis**

(GCH) $2^\kappa = \kappa^+$ for all infinite κ

all cardinal powers are determined, and assume the least consistent value, namely

Corollary 1 ((GCH))**.**
$$\kappa^\nu = \begin{cases} \kappa & \textit{if} \ \ \nu < \mathrm{cof}\,\kappa, \\ \kappa^+ & \textit{if} \ \ \kappa > \nu \geq \mathrm{cof}\,\kappa, \\ \nu^+ & \textit{if}\ \nu \geq \kappa. \end{cases}$$

Proof.

By induction on κ, by applying Buchowski's Theorem. □

GCH being notoriously (almost) totally independent on regular cardinals, one formulated the **Singular Cardinals Hypothesis (SCH)** $2^{\mathrm{cof}\,\kappa} < \kappa \implies \kappa^{\mathrm{cof}\,\kappa} = \kappa^+$ for all singular κ

Assuming (SCH), all cardinal powers are determined, and assume the least values consistent with the powes 2^ν of the regular cardinals ν, namely

Corollary 2 ((SCH))**.**

(i) for all κ, ν

$$\kappa^\nu = \begin{cases} 2^\nu & \text{if } \kappa \leq 2^\nu \ \ (in\ part.\ if\ \nu \geq \kappa), \\ \kappa & \text{if } \nu < \mathrm{cof}\,\kappa \ \ and\ \ 2^\nu < \kappa, \\ \kappa^+ & \text{if } \kappa > \nu \geq \mathrm{cof}\,\kappa \ \ and\ \ 2^\nu < \kappa. \end{cases}$$

(ii) for singular ν

$$2^\nu = \begin{cases} 2^{<\nu} & \text{if } \exists \kappa < \nu\ 2^\kappa = 2^{<\nu}, \\ (2^{<\nu})^+ & \text{otherwise.} \end{cases}$$

Proof. (i) Again by induction on κ, by applying Buchowski's Theorem.

(ii) Apply the Buchowski-Hechler theorem: the first case is immediate, while, when 2^κ is not eventually constant below ν, then $\mathrm{cof}\,\nu = \mathrm{cof}\,(2^{<\nu})$ and so, being ν singular and $2^{2^{\mathrm{cof}\,\nu}} = 2^{\mathrm{cof}\,\nu} < 2^{<\nu}$, SCH applies and gives $\gimel(2^{<\nu}) = (2^{<\nu})^+$. $\square$

Tarski's theorem on products

Theorem 3 (Tarski). *Let* ν *be an infinite cardinal, and let the* ν*-sequence of cardinals* $\langle \kappa_\alpha \mid \alpha < \nu \rangle$ *be weakly increasing,* i.e. *s.t.* $0 < \kappa_\alpha \leq \kappa_\beta$ *for* $\alpha < \beta < \nu$. *Then*

$$\prod_{\gamma<\nu} \kappa_\gamma = (\sup_{\gamma<\nu} \kappa_\gamma)^\nu. \tag{e9}$$

Proof. Put $\kappa = \sup_{\gamma<\nu} \kappa_\gamma$, so $\kappa \leq \prod_{\gamma<\nu} \kappa_\gamma$.
Let $\mathcal{P}$ be a partition ν in ν parts, such that κ is the supremum of the κ_γ on each part $X \in \mathcal{P}$. Then

$$\kappa^\nu \leq \prod_{\gamma<\nu} \kappa_\gamma^\nu = \prod_{X\in\mathcal{P}} \prod_{\gamma\in X} \kappa_\gamma^\nu \leq \prod_{X\in\mathcal{P}} (\sup_{\gamma\in X} \kappa_\gamma)^{|X|} = ((\sup_{\gamma<\nu} \kappa_\gamma)^\nu)^\nu = \kappa^\nu$$

□

Remark that the conditions of *weak monotonicity* and of *cardinal length* are always separately satisfiable, but not both together, in general.

Shelah's pcf theory

Let $a \subseteq Reg$ be a set of regular cardinals, which we assume to be an interval $[\aleph_\alpha, \aleph_\delta) \cap Reg$ of length $|a| < \aleph_\alpha$. Define

$$pcf(a) = \{\mathrm{cof}\,(\prod_{\kappa \in a} \kappa/\mathcal{D}) \mid \mathcal{D} \textit{ ultrafilter on } a\}, \text{ and}$$

$$pcf_\mu(a) = \bigcup\{pcf(b) \mid b \subseteq a, |b| \leq \mu\}, \text{ for } \mu \leq |a|$$

Lemma 4. *For all $\mu \leq |a|$:*

1. $a \subseteq pcf_\mu(a)$, *and* $\sup pcf_\mu(a) \leq (\sup a)^\mu$*;*
2. $\min pcf_\mu(a) = \min a$*.*

Proof.

1. For each $\kappa \in a$ take the principal ultrafilter generated by $\{\kappa\}$. On the other hand, $\mathrm{cof}\,(\prod_{\kappa \in b} \kappa/\mathcal{D}) \leq |\prod_{\kappa \in b} \kappa| \leq (\sup b)^{|b|}$.
2. All $\kappa \in a$ are regular, hence no sequence in $\prod_{\kappa \in a} \kappa$ of length less than $\min a$ can be cofinal modulo $\mathcal{D}$. □

The following theorems are the essential part of Shelah's pcf theory (so their proofs are elementary, but very complicated, and we omit them).

Let $a = [\aleph_\alpha, \aleph_\delta) \cap Reg$ and $\mu \leq |a| < \aleph_\alpha$. Then

Theorem 4. *$pcf_\mu(a) = [\aleph_\alpha, \aleph_\gamma] \cap Reg$, with $\aleph_\gamma$ regular $\geq \aleph_\delta$ and $|\gamma \setminus \alpha| \leq |\delta \setminus \alpha|^\mu$.*

Theorem 5. *If $\kappa^\mu < \aleph_\alpha$ for all $\kappa < \aleph_\alpha$, then $\aleph_\gamma = \aleph_\delta^\mu$.*

Theorem 6. *$|pcf_\mu(a)| \leq |a|^{+++} \leq |\delta|^{+++}$.*

Recall that $a = [\aleph_\alpha, \aleph_\delta) \cap Reg$ and $\mu \leq |a| < \aleph_\alpha$.

Corollary 3. *Let δ be limit. Then*

$$\kappa^\mu < \aleph_\alpha \ for \ all \ \kappa < \aleph_\alpha \implies \aleph_\delta^\mu < \aleph_{\alpha+|pcf(a)|^+},$$

hence
$$\mu < \aleph_\delta \implies \aleph_\delta^\mu < \aleph_{(|\delta|^\mu)^+}.$$

In particular, when $\aleph_\delta$ is a singular strong limit cardinal, then

$$2^{\aleph_\delta} = \beth(\aleph_\delta) < \aleph_{(2^{|\delta|})^+}.$$

Corollary 4. *In general, for all limit ordinal δ:*

$$\beth(\aleph_\delta) \leq {\aleph_\delta}^{|\delta|} < \max\{\aleph_{|\delta|^{++++}}, (2^{|\delta|})^+\}$$

Proof of Corollary 3.

The first assertion follows from theorems 4-5, because $pcf_\mu(a)$ is the interval $[\aleph_\alpha, \aleph_\delta^\mu] \cap Reg$, hence the index γ of $\aleph_\delta^\mu$ is an ordinal smaller than $|pcf_\mu(a)|^+$.

For the second assertion we distinguish four cases:

1. $\delta = \aleph_\delta$: then $|\delta|^\mu = \aleph_\delta^\mu < \aleph_{(|\delta|^\mu)^+}$;
2. $\aleph_\delta \leq 2^{|\delta|}$: then $\aleph_\delta^\mu = 2^\mu \leq |\delta|^\mu < \aleph_{(|\delta|^\mu)^+}$;
3. $\forall \kappa < \aleph_\delta\ (\kappa^\mu < \aleph_\delta)$: put $\aleph_\alpha = (|\delta|^\mu)^+$, so the first assertion gives $\aleph_\delta^\mu < \aleph_{\alpha + |pcf(a)|^+}$, and $|pcf(a)|^+ \leq (|a|^\mu)^+ \leq (|\delta|^\mu)^+$;
4. $\exists \kappa < \aleph_\delta\ (\kappa^\mu \geq \aleph_\delta)$: let $\aleph_\beta$ the least such κ, so for $\aleph_\beta$ case 3. holds, and one gets $\aleph_\delta^\mu = \aleph_\beta^\mu < \aleph_{(|\beta|^\mu)^+} \leq \aleph_{(|\delta|^\mu)^+}$.

□

Proof of Corollary 4.

If $\aleph_\delta \leq 2^\delta$ the estimate is obvious.

Otherwise let $\aleph_\alpha = (2^{|\delta|})^+ < \aleph_\delta$.Then put $\mu = |\delta|$ and apply corollary 3, obtaining $\aleph_\delta^{|\delta|} < \aleph_{\alpha+|pcf(a)|^+}$.

Now theorem 6 gives $|pcf(a) \leq |a|^{+++} \leq |\delta|^{+++}$, hence $\alpha + |pcf(a)|^+ \leq |\delta|^{++++}$, because and $\alpha < \delta$. $\square$

A remarkable consequence is the stunning estimate

$$2^{\aleph_0} < \aleph_\omega \implies \aleph_\omega^{\aleph_0} < \aleph_{\omega_4}$$