# References

1. C. C. Chang, H. J. Keisler - *Model Theory*, Elsevier 1998: 1.3, 2.1

2. T. Jech - *Set theory*, Academic Press 1978: Ch. II. 10-11

# Transitive closure

**Definition.** *A set* $X$ *is* transitive *if* $y \in x \in X \implies y \in X$.

*The transitive closure* $TC(X)$ *of the set* $X$ *is the set*

$$TC(X) = \bigcap\{T \ \textit{transitive} \mid X \subseteq T\} = \bigcup_{n \in \omega} X_n,$$

*where* $X_0 = X, \quad X_{n+1} = \cup X_n$.

**Remark 1.** *1. The definition is chosen so as to have*

$$X \text{ transitive} \iff X = TC(X).$$

*2. It is sometimes replaced by*

$$TC'(X) = \cap\{T \ \textit{transitive} \mid X \in T\} = TC(X) \cup \{X\} = TC(\{X\},$$

*so as to have* $X \neq TC'(X)$ *, and*

$$X \text{ transitive} \iff X \in TC'(X).$$

*3. An* $\in$*-inductive definition is*

$$TC(X) = X \cup \bigcup_{x \in X} TC(x), \quad \textit{or} \quad TC'(X) = \{X\} \cup \bigcup_{x \in X} TC'(x).$$

# Hereditary cardinal

**Definition.** *The hereditary cardinal of $X$ is the cardinal*

$$HC(X) = |TC(X)|$$

**Remark 2.**

1. $HC(X) = |TC(X)| = |TC'(X)|$ *whenever infinite;*
2. $HC(X) \leq \kappa \implies |\rho(X)| \leq \kappa$*;*
3. $HC(X) \leq \kappa \iff |X| \leq \kappa$ *and* $\forall x \in TC(X)\,(|x| \leq \kappa)$*;*
4. *if $\kappa$ is regular, then*

   $$HC(X) < \kappa \;\&\; \rho(X) < \kappa \iff |X| < \kappa \;\&\; \forall x \in TC(X)(|x| < \kappa).$$

# Transitive heritage of cardinals

**Definition.** *The heritage of the infinite cardinal* $\kappa$ *is the set*

$$H(\kappa) = \{X \mid HC(X) < \kappa\}.$$

**Remark 3.**

1. $H(\kappa)$ *is transitive, and* $H(\kappa) \subseteq V_\kappa$*;*

2. $H(\omega) = V_\omega$*, and* $H(\kappa) = V_\kappa$ *if* $\kappa$ *is inaccessible.*

# Natural models

1. $V_\alpha \models$ Ext, Fond, Sep, Union, AC, for all $\alpha > 0$;
2. $V_\alpha \models$ Pair, Pow $\iff$ $\alpha$ is limit;
3. $V_\alpha \models$ Infinity $\iff$ $\alpha > \omega$;
4. $V_\alpha \models$ ZFC $\Longleftarrow$ $\alpha$ is inaccessible (but not viceversa);

5. $\bigcup_{\alpha \in Ord} V_\alpha \models$ ZFC is provabble in ZF.

6. $H(\omega) = V_\omega \models$ ZFC $- Infinity$;
7. $H(\kappa) \models$ ZFC $- Pow \,\&\, Repl$ $\iff$ $\kappa > \omega$;
8. $H(\kappa) \models$ ZFC $- Pow$ $\iff$ $\kappa > \omega$ is regular;
9. $H(\kappa) \models$ ZFC $- Repl$ $\iff$ $\kappa > \omega$ is strong limit;
10. $H(\kappa) = V_\kappa \models$ ZFC $\iff$ $\kappa > \omega$ is (strongly) inaccessible.

# Mostowski's collapse

**Theorem 1.** *(Mostowski) Let* $E \subseteq A \times A$ *be an extensional wellfounded relation on* $A$*. Then there exists a unique transitive* $T$ *and a unique bijective* $\pi : A \to T$ *s.t.* $aEb \iff \pi(a) \in \pi(b)$ *for all* $a, b \in A$*. Moreover, if* $S \subseteq A$ *is transitive and* $E \cap (S \times S) = \in_{|S}$*, then* $S \subseteq T$ *and* $\pi_{|S} = id_S$*.*

**Proof.** If it exists, $\pi$ is unique because $\pi(a) = \{\pi(b) \mid bEa\}$, but this equality gives a good definition of $\pi$, by wellfoundedness of $E$: the subset $\{a \in A \mid$ the definition is not valid or not unique—$\}$ cannot have a minimal element. Moreover $\pi$ is injective because $E$ is extensional. Finally the identity is the unique $\in$-isomorphic embedding of a transitive set into a transitive superset. $\square$

# Transitive models

Let $M$ be a transitive set (or class). Then

1. $M \models Y = \mathcal{P}(X) \iff Y = \mathcal{P}(X) \cap M$;
2. $M \models Y = V_\alpha \iff Y = V_\alpha \cap M$;
3. $M \models |X| = |Y| \implies |X| = |Y|$ (but not viceversa);
4. $M \models \alpha \in Ord \iff \alpha \in Ord \cap M$;
5. $\alpha \in Card, \alpha \in Reg, \alpha \in LimCard, \alpha \ weakly \ inac \implies$ $M \models \alpha \in Card, \alpha \in Reg, \alpha \in LimCard, \alpha \ weakly \ inac,$ (but not viceversa);
6. $M \models \beta = |\alpha|, \ \beta = \operatorname{cof} \alpha \implies \beta \geq |\alpha|, \ \beta \geq \operatorname{cof} \alpha,$ (but strict inequality may hold);
7. $M \models$ ZFC $\impliedby \big(\alpha \ inaccessible \implies M \models \alpha$ inaccessible, (AC is needed to grant that $M \models either 2^\kappa \geq \alpha \ or \ 2^\kappa < \alpha$).

**Proposition 4.** *Let $M, N$ be transitive models of ZFC. Then $M = N$ if and only if $M$ and $N$ have the same sets of ordinals.*

**Proof.** The direct implication is obvious.

Now $M = N$ implies that they have the same ordinals, and also the same pairs of ordinals.

For all $X \in M$ exists in $M$ a bijection $\sigma$ between $TC'(X)$ and some ordinal $\alpha$. Define the relation $E$ on $\alpha$ by $\beta E \gamma \iff \sigma(\beta) \in \sigma(\gamma)$, and apply Mostowski's collapsing theorem (in the model N). Then $\pi(TC'(X) = TC'(X) \in N$, hence $X \in N$.

Then $N \subseteq M$ follows by $\in$-induction, and the converse inclusion is proven similarly. $\square$

# Recall First Order Logic

- language $\mathcal{L} = \{c_i; R_j; F_k\}$ symbols of constants, relations, functions;
- logic simbols $\neg, \vee, \wedge, \rightarrow, \leftrightarrow$; $\exists, \forall$; $=$;
- terms, formulas;
- sentences, theories; provability $T \vdash \Sigma$;
- assignments, structures and models; satisfaction $\mathcal{M} \models \phi[f]$;
- Skolem function $f$ for $\phi$: $\exists y(\phi(y, \overline{a}) \Longrightarrow \phi(f(\overline{a}), \overline{a})$

# Conseguenza logica

**Definition.** *si dice che l'enunciato $\tau$ è conseguenza logica della teoria (insieme di enunciati) $\Sigma$, scritto $\Sigma \vdash \tau$, se esiste una dimostrazione di $\tau$ in $\Sigma$.*

- *una dimostrazione di $\tau$ in $\Sigma$ è una successione di finita di formule $\langle \phi_i \mid 1 \leq i \leq n \rangle$ t.c. $\tau = \phi_n$, ed ogni $\phi_i$ è un assioma logico, oppure un elemento di $\Sigma$, oppure si ottiene da una o due precedenti formule mediante una regola di deduzione.*
- *le regole di deduzione sono*
  - *Modus ponens: da $\phi$ e $\phi \to \psi$ deduco $\psi$, e*
  - *generalizzazione: da $\phi$ deduco $\forall x \phi$, se $x$ non è libera in $\phi$.*
- *gli assiomi logici sono*
  - *tutte formule ottenute dalle tautologie proposizionali,*
  - $x = x$, $x = y \to y = x$, $x = y \wedge y = z \to x = z$, *e*
  - *Leibniz Rule:* $x = y \to (\phi(x) \leftrightarrow \phi(y))$ *per ogni formula* $\phi$.