References

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Reflexion principle

A set (or class) M *reflects the formula* ϕ (with the variables $x_1, \ldots, x_n$ free) if for all $a_1, \ldots, a_n \in M$

$$M \models \phi[a_1, \ldots, a_n] \iff \phi(a_1, \ldots, a_n) \textit{ is true } (\textit{in } V)$$

Equivalently $\forall a_1, \ldots, a_n \in M(\phi[a_1, \ldots, a_n] \iff \phi^M[a_1, \ldots, a_n])$, where ϕ^M is obtained from ϕ by restricting all quantifiers to M.

Theorem (Montague-Levy). *For all formula ϕ with $x_1, \ldots, x_n$ free, and all ordinal α, there exists $\beta > \alpha$ s.t. V_β reflects ϕ.*

Corollary. *For all A there exists $M \supseteq A$ s.t.* $|M| \leq \max(|A|, \aleph_0)$, *that reflects ϕ.*

Corollary. *On can reflect simultaneously any* finite *set of formulas (but not, in general, an infinite set).*

Proof. Write $\phi \equiv Q_1 y_1 \ldots Q_m y_m(\psi(y_1, \ldots, y_m; x_1, \ldots, x_n))$ in prenex form, let $\phi_r \equiv Q_{r+1} y_{r+1} \ldots Q_m y_m(\psi(y_1, \ldots, y_m; x_1, \ldots, x_n))$, and put

$$g_r(\overline{y}; \overline{x}) = \begin{cases} \text{min } \{\delta > \alpha \mid \exists y_r \phi_r(\overline{y}; \overline{x}) \to (\exists y_r \in V_\delta)\phi_r(\overline{y}; \overline{x})\} & \text{if } Q_r = \exists \\ \text{min } \{\delta > \alpha \mid \neg\forall y_r \phi_r(\overline{y}; \overline{x}) \to (\exists y_r \in V_\delta)\phi_r(\overline{y}; \overline{x})\} & \text{if } Q_r = \forall \end{cases}$$

$f_1(\gamma) = \max_{1 \le r \le m}(\sup\{g_r(\overline{y}; \overline{x}) \mid \overline{y}, \overline{x} \in V_\gamma\} + 1)$,

$f_{h+1}(\gamma) = f_1(f_h(\gamma)), \quad f_\omega(\gamma) = \sup_{h<\omega} f_h(\gamma)$.

Then $g_r(\overline{y}; \overline{x}) < f_\omega(\gamma)$ for all r and all $\overline{y}, \overline{x} \in V_{f_\omega(\gamma)}$, hence

$$Q_r y_r(\phi_r(\overline{y}; \overline{x})) \iff (Q_r y_r \in V_{f_\omega(\gamma)})(\phi_r^{V_{f_\omega(\gamma)}})(\overline{y}; \overline{x})).$$

Now the formula $\psi = \phi_m$ has no quantifiers, hence $\phi_m \equiv \phi_m^{V_{f_\omega(\gamma)}}$, and so, descending by induction on r from m to 0, one gets that $\phi_r \iff \phi_r^{V_{f_\omega(\gamma)}}$ for $m \ge r \ge 0$.

Finally ϕ_0 is ϕ, and by putting $\beta = f_\omega(\alpha)$, one obtains

$$\forall a_1, \ldots, a_n \in V_\beta\,(\phi[a_1, \ldots, a_n] \iff \phi^{V_\beta}[a_1, \ldots, a_n]),$$

□

Levy's theorem

Theorem (Levy). *Let ϕ be a Σ_1-formula with $x, x_1, \ldots, x_n$ free, and let κ be an uncountable cardinal.Then for all $a_1, \ldots, a_n \in H(\kappa)$*

$$\exists x\,(\phi(x, a_1, \ldots, a_n)) \implies \exists x \in H(\kappa)(\phi(x, a_1, \ldots, a_n)).$$

Corollary . *Any term that "increases cardinality" (like $\mathcal{P}(x)$, $\aleph(x), {}^y x$) cannot be Σ_1^{ZFC}.*

Proof. Remark that we may assume w.l.o.g. that $\kappa = \mu^+$ is a successor cardinal, because, if κ is a limit cardinal, then

$$a_1, \ldots, a_n \in H(\kappa) \iff \exists \mu < \kappa\,(a_1, \ldots, a_n \in H(\mu^+)).$$

Moreover we may assume that the formula ϕ is Σ_0, because, if ϕ is $\exists y\,(\psi(y, x, \overline{x})$ with $\psi \in \Sigma_0$, then

$\exists x\, \phi(x, \overline{a}) \Leftrightarrow \exists z\,(z = \{x, y\} \wedge \psi(y, x, \overline{a}))$,and $z \in H(\kappa) \Rightarrow x, y \in H(\kappa)$.

Now assume that given $a_1, \ldots, a_n \in H(\kappa)$ there exists a s.t. $\phi(a, \overline{a})$. Put $B = \{a\} \cup TC(\{a_1, \ldots, a_n\})$, so $|B| < \kappa$, and pick β s.t. $V_\beta \supseteq B$ and reflects ϕ, so $V_\beta \models \phi(a, \overline{a})$. Close the set B under a countable family of Skolem functions for V_β, to get an elementary submodel C of V_β of size $|C| = |B| \cdot \aleph_0 < \kappa$.

Let $\pi : C \to T$ be the Mostowski collapse (that works because the structure $(C, \in_{|C \times C})$ is extensional and wellfounded): then $T \in H(\kappa)$, $\pi_{|TC(\{a_i\})}$ is the identity, hence $a_i \in T$, and $T \models \phi(\pi(a), \overline{a})$. Finally, $\phi \in \Sigma_0$ is absolute for transitive models, hence $\pi(a)$ is the wanted element of $H(\kappa)$. $\square$

Gödel-Bernays class theory GB

Same language $\{\in\}$ of ZF, intended objects are *classes* (denoted by capitals); *sets* (usually denoted by lower case letters) are those classes *that belong to some class*: $Set(x) \Leftrightarrow \exists X(x \in X)$.

The finite set of axioms of GB consists of five groups:

A. General axioms

A.1 Extensionality: $\forall x(x \in A \Leftrightarrow x \in B) \Longrightarrow A = B)$

A.2 Pair: $\forall x, y\, \exists z\, (z = \{x, y\})$.

B. Class axioms:

B.1 Membership: $\exists E = \{(x, y) \mid x \in y\}$;

B.2 Intersection: $\forall A, B\, \exists C = A \cap B$;

B.3 (Absolute) Complement: $\forall A\, \exists B = V \setminus A = \{x \mid x \notin A\}$;

B.4 Cartesian Product: $\forall A\, \exists B = A \times V = \{(x, y) \mid x \in A\}$;

B.5 Domain: $\forall A\, \exists B = \text{dom}\ A = \{x \mid \exists y\, (x, y) \in A\}$;

B.6 Cyclic Permutation: $\forall A\, \exists B = \{(x,y,z) \mid (y,z,x) \in A\}$;

B.7 Transposition: $\forall A\, \exists B = \{(x,y,z) \mid (x,z,y) \in A\}$.

C. Set axioms:

C.1 Infinite: $\exists x\, (\emptyset \in x \ \wedge \forall y (y \in x \Rightarrow y \cup \{y\} \in x))$;

C.2 Union: $\forall x\, (\bigcup x = \{z \mid \exists y \in x\, (z \in y)\} \in V)$;

C.3 Powerset: $\forall x\, (\mathcal{P}(x) = \{y \mid y \subseteq x\} \in V)$;

C.4 Image: F univalent $\Rightarrow \forall x\, (F[x] = \{t \mid \exists s \in x\, ((s,t) \in F)\} \in V)$;

N.B. $\bigcup X, \mathcal{P}(X), F[X]$ exist for any class X, but may be *proper*.

D. Foundation: $\forall X \neq \emptyset\, \exists x \in X\, (x \cap X = \emptyset)$.

E. Universal Choice: $\exists F : V \to V\, \forall x \neq \emptyset (F(x) \in x)$.

Theorem. *Let ϕ be a formula with all quantifiersl restricted to V, whose free variables are among $x_1, \ldots, x_n, X_1, \ldots, X_m$: then, for all $A_1, \ldots, A_m$, there exists the class*

$$\{(x_1, \ldots, x_n) \mid \phi(x_1, \ldots, x_n, A_1, \ldots, A_m)\}.$$

GB vs. ZF

Clearly $ABCD \vdash V \models$ ZF, and $ABCDE \vdash V \models$ ZFC.

More interesting is the followuing

Theorem . *The theories ABCD and ABCDE are consevative extensions of ZF and ZFC, respectively,* i.e. *they prove exactly the same theorems that involve only sets.*

Proof. Let $\mathfrak{M} = (M, R)$ be a model of ZF, and add to M all subsets of M obtained by taking M for V, R for E, and closing under the operations B.2-B.7. Identify the element $x \in M$ with the "class" $\{t \in M \mid tEx\}$: then one obtains a model $\mathfrak{N}$ of ABCD with exactly the same sets as M.

It follows that if $\mathfrak{M} \models$ ZFC, then $\mathfrak{N} \models AC$, but not necessarily full GB, because E postulates a proper class function: only by forcing one gets a suitable generic extension $\mathfrak{N}' = \mathfrak{N}[G] \models E$.

□