# References


1. T. Jech - *Set theory*, Academic Press 1978: Ch. II. 10-14

2. F. Drake - *Set theory*, North Holland 1974: Ch. V

# Definable sets

**Definition.** *The set* $X$ *is a definable subset of* $A$ *if there exist a formula* $\phi(x, x_1, \ldots, x_n)$ *and elements* $a_1, \ldots, a_n \in A$ *s.t.*

$$X = \{a \in A \mid A \models \phi[a, a_1, \ldots, a_n]\}.$$

**Lemma 1.**

1. $A \in Def(A)$ *and* $\mathcal{P}_\omega(A) \subseteq Def(A)$*;*
2. $A$ *transitive implies* $Def(A)$ *transitive.*

**Proof.** 1. $A = \{a \in A \mid a = a\}$, so the formula $x = x$ defines $A$, and $x = a_1 \vee \ldots \vee x = a_n$ defines the finite subset $\{a_1, \ldots, a_n\} \subseteq A$.
2. Let $b \in B = \{b \in A \mid A \models \phi[b, \overline{a}]\} \in Def(A)$: if $A$ is transitive, then $b \subseteq A$, so $b = \{a \in A \mid a \in b\}$, with parameter $b$ in $A$. $\square$

# The constructible hierarchy $L_\alpha$

$L_0 = \emptyset, \quad L_{\alpha+1} = Def(L_\alpha), \quad L_\lambda = \bigcup\{L_\alpha \mid \alpha < \lambda\}$ for limit $\lambda$.

By induction on $\alpha$ one proves

**Lemma 2.** *1. $L_\alpha$ is transitive, and $\alpha < \beta \implies L_\alpha \in L_\beta$;*

*2. $\mathcal{P}_\omega(A) \subseteq Def(A) \subseteq \mathcal{P}(A) \implies \forall\alpha(L_\alpha \subseteq V_\alpha)$ and $\forall n(L_n = V_n)$;*

*3. $|L_\alpha| = |\alpha|$ for $\alpha \geq \omega$;*

*4. $\alpha = \{a \in L_\alpha \mid L_\alpha \models a \textit{ is an ordinal}\} \in Def(L_\alpha) = L_{\alpha+1}$;*

*5. for $x \in L$, $\delta(x) = \min\{\alpha \mid x \in L_\alpha\} > \rho(x)$ is a successor;*

*6. $x \in y \in L \implies \delta(x) < \delta(y)$.* □

**Lemma 3.** *(covering lemma) If $x \subseteq L$, then $\exists\alpha\,(x \subseteq L_\alpha)$.*

**Proof.** *Put $\alpha = \sup\{\delta(y) \mid y \in x\}$: then $x \subseteq L_\alpha$.* □

## ZF $\vdash$ "$L \models$ ZF $+ V = L$"

**Proof.** $L$ is a transitive class containing all ordinals, so

•$L \models Ext, Fond, Inf$;

•$L \models Pair$: for $a, b \in L_\alpha$, $\{a, b\} = \{x \in L_\alpha \mid L_\alpha \models x = a \vee x = b\}$;

•$L \models Un$: for $a \in L_\alpha$, $\cup a = \{x \in L_\alpha \mid L_\alpha \models \exists y \in a(x \in y)\}$;

•$L \models Pow, Repl$: if $f$ is a function and $a \in L_\alpha$, some $L_\beta$ includes both $\mathcal{P}(a) \cap L$ and $f[a] \cap L$, so they can be obtained by Sep.

•$L \models Sep$: given a formula $\phi$ and $a \in L_\alpha$, pick $\beta > \alpha$ s.t. $V_\beta$ reflects the formulae $x \in L\gamma$, $x \in L \equiv \exists\delta(x \in L_\delta)$, $\phi^L$. Then

1. $\forall x \in V_\beta \forall\gamma < \beta(x \in L_\gamma \Leftrightarrow x \in L_\gamma^{V_\beta})$,
2. $\forall x \in V_\beta(x \in L \Leftrightarrow \exists\delta < \beta(x \in L_\delta^{V_\beta}))$,
3. $\forall x \in V_\beta(\phi^L(x) \Leftrightarrow (\phi^L)^{V_\beta}(x) \Leftrightarrow \phi^{L \cap V_\beta}(x))$.

By 1-2, $\forall x \in V_\beta(x \in L \Leftrightarrow \exists\delta < \beta(x \in L_\delta))$, but $a \in L_\beta = L \cap V_\beta$, and point 3 gives $\{x \in a \mid \phi^L(x)\} = \{x \in L_\beta \mid \phi^{L_\beta}(x) \wedge x \in a\} =$
$= \{x \in L_\beta \mid L_\beta \models \phi(x) \wedge x \in a\} \in Def(L_\beta) = L_{\beta+1}$.

•$L \models V = L$: $L_\alpha$ is absolute between transitive models of ZF, being $\Delta_1^{ZF}$, hence $L_\alpha^L = L_\alpha$, and $L^L = L$. □

# universal wellordering and choice



Define inductively a wellorder $\prec_\alpha$ on $L_\alpha$ by

$x \prec_{\alpha+1} y$ if

- $x \in L_\alpha$ and $y \in L_{\alpha+1} \setminus L_\alpha$, or $x, y \in L_\alpha$ and $x \prec_\alpha y$, or
- both $x, y \in L_{\alpha+1} \setminus L_\alpha$, and
  - the first formula (in the ordering of $V_\omega$) that defines $x$ on $L_\alpha$ precedes the first one that defines $y$, or
  - the first formula is the same, and the first tuple of parameters (in the lexicographic ordering of $L_\alpha^n$) that defines $x$ precedes the first one that defines $y$.

- $\alpha < \beta \implies L_\alpha$ initial segment of $L_\beta$, so $\prec_L = \bigcup\{\alpha \in Ord \mid \prec_\alpha\}$ is a definable wellordering of the class $L$.
- The definition of $\prec_\alpha$ is $\Delta_1^{ZF}$, so $x <_L y \equiv \exists\alpha(x <_\alpha y)$ is $\Sigma_1^{ZF}$, but if $V = L$ then $x <_L y \Leftrightarrow \forall\alpha(x <_\alpha y)$, too, hence $\prec_L$ is $\Delta_1^{ZF+V=L}$

# Relative consistency of AC and GCH

**Lemma 4.** $\forall\kappa \geq \omega\,(\,L \cap H(\kappa) = L_\kappa)$.

**Proof.** $x \in L$ is $\Sigma_1^{ZF}$, so if $\kappa > \omega$ Levy's theorem gives

$$\exists\alpha(x \in L_\alpha) \iff \exists\alpha \in H(\kappa)(x \in L_\alpha).$$

While for $\kappa = \omega$ one has $H(\omega) = V_\omega = L_\omega$. □

**Lemma 5.** $\forall\kappa \geq \omega\,(L \cap \mathcal{P}(\kappa) \subseteq L_{\kappa^+})$.

**Proof.** $\mathcal{P}(\kappa) \subseteq H(\kappa^+)$, hence $L \cap \mathcal{P}(\kappa) \subseteq L \cap H(\kappa^+) = L_{\kappa^+}$. □

**Theorem 1.** *V=L $\implies$ GCH, hence $L \models GCH$.*

**Proof.** If $L$ is the whole universe, then all cardinals are preserved, and $\kappa < 2^\kappa \leq |L_{\kappa^+}| = \kappa^+$. □