References

1. T. Jech - *Set theory*, Academic Press 1978: Ch. III. 18

Boolean Models

Let U be a transitive set (or class), and let B be a *complete Boolean algebra*. A *Boolean model* of set theory with universe U and values in B is a 4-tuple $(U, B; I, E)$ where the evaluating functions $I, E : U^2 \to B$ assign values $I(x, y) = \| x = y \|_B \in B$ and $E(x, y) = \| x = y \|_B \in B$ to the atomic formulae $x = y$ and $x \in y$, resp., subject to the follwing conditions, for all $x, y, z, v, w \in U$:

1. $\| x = x \|_B = 1$;
2. $\| x = y \|_B = \| y = x \|_B$;
3. $\| x = y \|_B \cdot \| y = z \|_B \leq \| x = z \|_B$;
4. $\| x \in y \|_B \cdot \| v = x \|_B \cdot \| w = y \|_B \leq \| v \in w \|_B$.

(Usually the subscript B is omitted.)

Evaluating formulae

By induction on the generation of ϕ, for $(\overline{x}) \in U^n$ put:

- $\| \neg\phi(\overline{x}) \|_B = - \| \phi(\overline{x}) \|_B$;
- $\| \phi(\overline{x}) \wedge \psi(\overline{x}) \|_B = \| \phi(\overline{x}) \|_B \cdot \| \psi(\overline{x}) \|_B$;
- $\| \phi(\overline{x}) \vee \psi(\overline{x}) \|_B = \| \phi(\overline{x}) \|_B + \| \psi(\overline{x}) \|_B$;
- $\| \phi(\overline{x}) \to \psi(\overline{x}) \|_B = -\| \phi(\overline{x}) \| + \| \psi(\overline{x}) \| = \| \phi(\overline{x}) \| \Rightarrow \| \psi(\overline{x}) \|$;
- $\| \exists x \phi(x, \overline{x}) \|_B = \sum_{x \in U} \| \phi(x, \overline{x}) \|_B$;
- $\| \forall x \phi(x, \overline{x}) \|_B = \prod_{x \in U} \| \phi(x, \overline{x}) \|_B$.

Remark that $\| \phi(\overline{x}) \to \psi(\overline{x}) \|_B = 1 \iff \| \phi(\overline{x}) \|_B \leq \| \psi(\overline{x}) \|_B$.

Call a formula $\phi(\overline{x})$ *valid for* $\overline{x} \in U$ if $\| \phi(\overline{x}) \|_B = 1$: then the *axioms of equality* and *all axioms of first order logic* are *valid*.

Hence *provably equivalent formulae receive equal values*.

Quotienting the algebra

Let Ψ be a *complete algebra homomorphism* of B onto B', and let $\mathcal{F}$ be the dual filter of the kernel of Ψ (an ideal of B).

Define the equivalence $\equiv_{\mathcal{F}}$ on U by $x \equiv_{\mathcal{F}} y \iff \| x = y \|_B \in \mathcal{F}$.

Let π be the projection of U onto the quotient $U' = U/\mathcal{F}$, and define I', E'' by $I' \circ \pi = \Psi \circ I$ and $E' \circ \pi = \Psi \circ E$.

Then $(U', B'; I', E')$ is a boolean model, called the quotient model of U modulo $\mathcal{F}$.

When B' is the trivial two-element algebra $\{0, 1\} = \mathbb{Z}_2$ one obtains a boolean quotient model corresponding to an *ordinary* (possibly nonstandard) model (M, R) satisfying

$$M \models \phi[\pi(\overline{u})] \iff \Psi(\| \phi(\overline{u}) \|_B) = 1.$$

Full models

More generally, call *full* a boolean model U if for all formula $\phi(x,\overline{y})$, with x free, and all $(\overline{v}) \in U^n$, there exists $u \in U$ s.t.

$$\| \exists x \phi(x,\overline{v}) \|_B = \sum_{x \in U} \| \phi(x,\overline{v}) \|_B = \| \phi(u,\overline{v}) \|_B .$$

Given an *ultrafilter* $\mathcal{F}$ on B, and a *full* Boolean model U, the ordinary *quotient model* $U/\mathcal{F} = (M,R)$ satisfies the following

Lemma.

Let $a_1,\ldots,a_n \in M$ *be the* $\equiv_F$*-classes of* $u_1,\ldots,u_n \in U$*. Then*

$$M \models \phi[a_1,\ldots,a_n] \iff \| \phi(u_1,\ldots,u_n) \|_B \in \mathcal{F}$$

NB: When the algebra B is *finite*, all Boolean Models are full, and all ultrafilters are principal.

The Boolean model V^B

Let B be a complete Boolean algebra, and define inductively

$$V_0 = \emptyset, \quad V^B_{\alpha+1} = \{f : X \to B \mid X \subseteq V^B_\alpha\}, \quad V^B_\lambda = \bigcup_{\alpha<\lambda} V^B_\alpha \text{ (limit } \lambda).$$

Let V^B be the class union of all V^B_α, and define the *Boolean rank* of $x \in V^B$ by $\rho^B(x) = \min \{\alpha \mid x \in V^B_\alpha\}$.

Inductively on the lexicografically ordered pairs $(\rho^B(x), \rho^B(y))$ define (recalling the boolean operation $a \Rightarrow b = -a + b$):

1. $\| x \in y \|_B = \sum_{t \in \text{dom } y} y(t) \cdot \| x = t \|_B$;
2. $\| x \subseteq y \|_B = \prod_{t \in \text{dom } x} (x(t) \Rightarrow \| t \in y \|_B)$;
3. $\| x = y \|_B = \| x \subseteq y \|_B \cdot \| y \subseteq x \|_B$.

Theorem. V^B *is a full Boolean model s.t.*

$$\| \forall t (t \in x \to t \in y) \|_B \leq \| x \subseteq y \|_B,$$

i.e. *the axiom of extensionality has Boolean value* 1.

Restricted quantification

Remark. *The following shortenings are usefull*

- $\parallel (\exists x \in y)\phi(x) \parallel = \Sigma_{t \in \text{dom } y}\, y(t) \cdot \parallel \phi(t) \parallel$: *in fact*

  $\|\exists x(x \in y \wedge \phi(x))\| = \Sigma_x(\|\phi(x)\| \cdot \Sigma_{t \in \text{dom } y}\, y(t) \cdot \parallel x = t \parallel) =$

  $= \Sigma_{t \in \text{dom } y}\, y(t) \cdot \Sigma_x(\|\phi(x)\| \cdot \|x = t\|) \leq \Sigma_{t \in \text{dom } y}\, y(t) \cdot \parallel \phi(t) \| \leq$

  $\leq \Sigma_t \, \|t \in y\| \cdot \|\phi(t)\| = \|\exists t(t \in y \wedge \phi(y))\|$.

- $\parallel (\forall x \in y)\phi(x) \parallel = \Pi_{x \in \text{dom } y}\, y(x) \Rightarrow \parallel \phi(x) \parallel$: *in fact*

  $\|\forall x(x \in y \rightarrow \phi(x))\| = \Pi_x((\Sigma_{t \in \text{dom } y}\, y(t) \cdot \|x = t\|) \Rightarrow \|\phi(x)\|) =$

  $\leq \Pi_{u \in \text{dom } y}((\Sigma_{t \in \text{dom } y}\, y(t) \cdot \|u = t\|) \Rightarrow \|\phi(u)\|) \leq$

  $\leq \Pi_{u \in \text{dom } y}(y(u) \Rightarrow \|\phi(u)\|) \leq \Pi_{u \in \text{dom } y}(\parallel u \in y \parallel \Rightarrow \parallel \phi(u) \parallel)$.

The canonical embedding

Definition . *The* canonical name $\breve{x} \in V^B$ *of the set* $x \in V$ *is defined by* $\in$*-induction:*

$$\text{dom}\, \breve{x} = \{\breve{y} \mid y \in x\} \quad and \quad checkx(\breve{y}) = 1 \quad for\ all\ \breve{y} \in \text{dom}\, \breve{x}.$$

Theorem. *Let* ϕ *be a* Δ_0*-formula and* $\overline{u} = (u_1, \ldots, U_n)$*. Then*

$$V \models \phi[u_1, \ldots, u_n] \iff \phi(\breve{u_1}, \ldots, \breve{u_n}) = 1.$$

Hence, if ϕ *is* Σ_1*, then* $V \models \phi[u_1, \ldots, U_n] \Longrightarrow \phi(\breve{u_1}, \ldots, \breve{u_n}) = 1$.
In particular

$$\| x \in Ord \| = \sum_{\alpha \in Ord} \| x = \breve{\alpha} \| \quad and \quad \| x \in L \| = \sum_{a \in L} \| x = \breve{a} \|$$

The main theorem

Theorem. *All axioms of ZFC have value* 1 *in* V^B.

Definition. *The* canonical name $\check{\mathbb{G}} \in V^B$ *for a generic ultrafilter on* B *is defined by:*

$$\text{dom}\,\mathbb{G} = \{\check{b} \mid b \in B\} \quad and \quad \mathbb{G}(\check{b}) = b \quad for\ all\ \check{b} \in \text{dom}\,\mathbb{G}.$$

Then

1. $\| \mathbb{G}\ ultrafilter\ on\ P(\check{B}) \| = 1$,
2. $\Pi_{\check{X} \in \mathcal{P}(\check{B})}(\| \check{X} \subseteq \mathcal{P}(\check{B}) \| \Rightarrow \| \Pi \check{X} \in \mathbb{G} \|) = 1$.

CAVEAT: $\mathbb{G}$ is not, in general, a complete ultrafilter in V^B, because $\mathcal{P}(\check{B})$ might be properly contained in $\mathcal{P}(\check{B})$. Similarly the algebra $\check{B}$ is $\mathcal{P}(\check{B})$-complete, but possibly not complete.