References

1. T. Jech - *Set theory*, Academic Press 1978: Ch. III. 16-17

2. P.J. Cohen - *La teoria degli insiemi e l'ipotesi del continuo*, Feltrinelli 1974: Ch. III

Generic extensions

Let $M \models$ ZFC be a transitive set (or class), and let P be a *notion of forcing in* M, *i.e.* a poset $P \in M$. The *generic extension* of M by the *generic set* G is a transitive $M[G] \supseteq M$ s.t.:

1. $M[G] \models$ ZFC;
2. $\emptyset \neq G \in M[G]$ is a M-*generic filter* on P, *i.e.*
   - $p \leq q \in G \implies q \in G$,
   - $\forall p, q \in G\ \exists r \in G\ (r \leq p, q)$, and
   - $(D \in M \cap \mathcal{P}(P)\,\&\,\forall p \in P\,\exists d \in D\,(d \leq p)) \implies D \cap G \neq \emptyset$;
3. $Ord^{M[G]} = Ord^{M}$, and
4. $(N\ transitive,\ M \subseteq N \models$ ZF, $G \in N) \implies M[G] \subseteq N$.

Forcing relations

The *forcing language* associated to $P \in M$ is

$$\mathcal{L} = \{=, \in; \mathcal{C}\} \text{ with } \mathcal{C} \supseteq \{\mathbb{G}\} \cup \{\, \check{\mathsf{x}} |\, x \in M\},$$

where $\mathbb{G}$ and the $\check{\mathsf{x}}$ are constant symbols, whose intended meaning are the generic set G and the element $x \in M$, resp.

The *forcing relation* $p \Vdash \sigma$ between *forcing conditions* $p \in P$ and *sentences* σ *of the forcing language* $\mathcal{L}$ is subject to appropriate logical rules, so as to obtain the following

Theorem. *(Forcing theorem)* $M[G] \models \sigma \iff \exists p \in G\; p \Vdash \sigma$.

In fact we shall have

(i) $p \Vdash \sigma \iff \forall G\, (p \in G \Rightarrow M[G] \models \sigma)$, and

(ii) $\forall p \in P\; \exists G \subseteq P\;\; M\text{-}generic\ s.t.\ p \in G$.

Forcing conditions

The poset P is in the ground model M, and the forcing relation must be defined in M.

Condition (ii) is always fulfilled when M is countable.

Condition (i) is verified provided that any element of $M[G]$ is the interpretation of a constant $c \in \mathcal{C}$, and the following conditions are fulfilled (assuming $\vee, \rightarrow, \exists$ defined through $\neg, \wedge, \forall$):

1. $p \Vdash \sigma \ \& \ q \leq p \implies q \Vdash \sigma$;
2. $\forall p \ \exists q \leq p. \ q \Vdash \sigma \vee q \Vdash \neg\sigma$;
3. $q \Vdash \sigma \implies q \nVdash \neg\sigma$;
4. $q \Vdash \sigma \wedge \tau \iff q \Vdash \sigma \ \& \ q \Vdash \tau$;
5. $q \Vdash \forall x \varphi(x) \iff q \Vdash \varphi(c)$ for all $c \in \mathcal{C}$.

Adding new reals

Example. In order to add new subsets of ω to the ground model M, the typical notion of forcing is

$$P = \{f : E \to \{0,1\} \mid E \in \mathcal{P}_\omega(\eta \times \omega)\}, \text{ with } p \le q \Leftrightarrow q \subseteq p.$$

For G M-generic, put $\chi = \bigcup G$. Then $M[G] \models \chi : \eta \times \omega \to \{0,1\}$, because the sets $D_{\alpha n} = \{p \in P \mid (\alpha, n) \in \text{dom } p\}$ are dense. Moreover, the sets $x_\alpha = \{n \in \omega \mid p(\alpha, n) = 1\} \subseteq \omega$ are different from one another and from any $x \in \mathcal{P}(\omega)^M$, because all sets $D_{\alpha\beta} = \{p \in P \mid \exists n \; p(\alpha, n) \neq p(\beta, n)\}$ are dense for $\alpha \neq \beta$, as are all sets $D_{\alpha x} = \{p \in P \mid \exists n \; p(\alpha, n) \neq \chi_x(n)\}$ for $x \in M$.

CAVEAT: even if η is a *cardinal in* M, it might be *collapsed in* $M[G]$ to $|\mathcal{P}(\omega)^M|^{M[G]}$.

Special posets

Let κ be a cardinal. The poset P is

1. κ-*closed* if every weakly descending κ-sequence in P is bounded from below in P;
2. κ-*distributive* if the intersection of κ open dense subsets of P is open dense in P;
3. κ-*saturated* if every subset of P whose elements are pairwise incompatible has size less than κ. The *saturation* $\mathrm{sat}(P)$ of P is the least κ s.t. P is κ-saturated.

Remark 1. *Any κ-closed poset is κ-distributive.*

Remark 2. *The saturation $\kappa = \mathrm{sat}(P)$ of the poset P is regular.*