Example: control theory [Datta, Ch. 5]

Control theory (important subject in engineering) is the study of dynamical systems + controllers.

Example can we keep an 'inverted pendulum' of length ℓ in the unstable upright position (12 o' clock) by applying a steering force?

State $x(t) = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}$, where θ is the angle formed by the pendulum (12 o' clock $\leftrightarrow$ $\theta = 0$).

Free system equations:

$$\dot{x} = \begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} x_2 \\ g\ell \sin x_1 \end{bmatrix} \approx \begin{bmatrix} x_2 \\ g\ell x_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ g\ell & 0 \end{bmatrix} x.$$

The system is not stable: $A = \begin{bmatrix} 0 & 1 \\ g\ell & 0 \end{bmatrix}$ has one positive and one negative eigenvalue.

Example: controlling an inverted pendulum

Now we apply an additional steering force u (control):

$$\dot{x} = Ax + Bu, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Can we choose $u(t)$ so that the system is stable? Yes — even better: we can choose one of the form $u(t) = Fx(t)$, $F \in \mathbb{R}^{1\times 2}$

We can literally build a contraption (engine + camera) that sets the appropriate force according to the current state only (feedback control). $u = \begin{bmatrix} f_1 & f_2 \end{bmatrix} x$ gives the closed-loop system

$$\dot{x} = (A + BF)x = \begin{bmatrix} 0 & 1 \\ f_1 + g\ell & f_2 \end{bmatrix} x.$$

Choosing f_1, f_2, we can move the eigenvalues of $A + BF$ arbitrarily.

Remark: $f_2 = 0$ (observing only position θ) isn't enough!

Other examples

Heat equation: in a bar of uniform material (the segment $[0, 1]$), one endpoint 1 is kept at constant temperature 0°C, and we apply a variable temperature (amount of 'heat') $u(t)$ at the other endpoint 0.
The temperature $x(y, t)$ at position y and time t follows

$$\frac{\partial}{\partial t} x(y,t) = \alpha \frac{\partial^2}{\partial y^2} x(y,t), \quad x(0,t) = u(t),\, x(1,t) = 0.$$

We discretize in space: $x(t)$ is a vector of temperatures at equi-spaced points $h, 2h, \dots, nh = 1$.

$$\frac{d}{dt} x(t) = Ax(t) + Bu(t),$$

$A = \frac{\alpha}{h^2} \operatorname{tridiag}(-1, 2, -1)$, $B = -\frac{\alpha}{h^2} e_1$.

Other examples in [Datta, Ch. 5], e.g. electrical circuits.

Video: triple pendulum on a cart, e.g., `youtu.be/cyN-CRNrb3E`.

The general setup

$$\dot{x} = Ax + Bu, \quad A \in \mathbb{R}^{n \times n},\, B \in \mathbb{R}^{n \times m}.$$

Q1 Can we stabilize the system around 0, i.e., choose $u(t) = Fx(t)$ so that $A + BF$ is stable?

Q2 Can we control the system, i.e., choose $u(t)$ to reach a given value of $x(t_F)$?

Not always: counterexample:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}.$$

No matter what we choose, we cannot change the dynamics of the second block of variables. If A_{22} has eigenvalues outside the LHP, there is nothing we can do.

Controllability

This structure may be 'hidden' behind a change of basis, for instance $A \leftarrow MAM^{-1}, B \leftarrow MB$.

How do we check for it? Krylov spaces:

The pair (A, B) is called controllable if

$$\operatorname{span}(B, AB, \dots, A^k B, \dots) = \mathbb{R}^n.$$

Controllability [Datta, Ch. 6, with more streamlined proofs]

Definition

$(A, B) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m}$ is controllable iff $K(A, B) = \mathbb{R}^n$, where

$$K(A, B) := \operatorname{span}(B, AB, A^2B, \dots).$$

It is enough to stop at $A^{n-1}B$, because A^n is a linear combination of $I, A, \dots, A^{n-1}$ (Cayley–Hamilton theorem).

Lemma

There exists a nonsingular $M \in \mathbb{R}^{n\times n}$ such that

$$M^{-1}AM = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \quad M^{-1}B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}$$

(with $A_{11} \in \mathbb{R}^{n_1\times n_1}$, $A_{22} \in \mathbb{R}^{n_2\times n_2}$, $B_1 \in \mathbb{R}^{n_1\times m}$, and $n_2 \neq 0$) if and only if (A, B) is not controllable.

Proof

$\Rightarrow$ Partition $M = \begin{bmatrix} M_1 & M_2 \end{bmatrix}$ conformably. Then,

$A^k B = M \begin{bmatrix} A_{11}^k B_1 \\ 0 \end{bmatrix} = M_1 A_{11}^k B_1$, so $K(A, B) \subseteq \operatorname{Im} M_1$.

$\Leftarrow$ Let the columns of M_1 be a basis of $K(A, B)$, and complete it to a nonsingular $M = \begin{bmatrix} M_1 & M_2 \end{bmatrix}$. Then, $M^{-1}AM$ is block triangular (because M_1 is A-invariant), and $M^{-1}B$ has zeros in the second block row (because the columns of B lie in $\operatorname{Im} M_1$).

(Linear algebra characterization: $K(A, B)$ is the smallest A-invariant subspace that contains B. It's the space V_n that we obtain after we encounter breakdown in Arnoldi.)

Kalman decomposition

Kalman decomposition

For every matrix pair $(A, B) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m}$, there is a change of basis M such that

$$M^{-1}AM = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \quad M^{-1}B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix},$$

with (A_{11}, B_1) controllable.

Proof: as above: take M_1 such that its columns are a basis of the 'controllable space' $K(A, B)$, then complete it to a basis of $\mathbb{R}^n$.

Other controllability criteria

Popov (or Hautus) criterion

(A, B) controllable $\iff$ $\operatorname{rank}[A - zI, B] = n$ for all $z \in \Lambda(A)$ $\iff$ $\operatorname{rank}[A - zI, B] = n$ for all $z \in \mathbb{C}$.

It is enough to test the condition on $z \in \Lambda(A)$, because for all other z we already have $\operatorname{rank}(A - zI) = n$.

Proof

$\Leftarrow$ If (A, B) is not controllable, write it in a Kalman decomposition, then for $z \in \Lambda(A_{22})$ the bottom part does not have full rank.

$\Rightarrow$ If $v^*[A - \lambda I, B] = 0$ for some $\lambda \in \Lambda(A)$, then up to a change of basis we can assume $v = e_n$, and this implies (A, B) are in a Kalman decomposition (with $n_2 = 1$).

Controllability Gramian

(A, B) controllable iff

$$W = \int_0^t \exp(\tau A) B B^* \exp(\tau A)^* \mathrm{d}\tau \succ 0$$

for $t > 0$ (one or all, equivalently).

Proof

$\Leftarrow$ suppose (A, B) is not controllable. Then, for any t
$\operatorname{Im} X \subseteq K(A, B)$, because $\operatorname{Im} \exp(\tau A) B x \in K(A, B)$.
$\Rightarrow$ suppose instead that for some $v \neq 0$ and $t > 0$

$$0 = v^* W v = \int_0^t v^* e^{At} B B^* e^{A^* t} v \mathrm{d}t \implies \Phi(t) = v^* e^{At} B \equiv 0.$$

Evaluate $0 = \Phi(0) = \Phi'(0) = \Phi''(0) = \dots$, we get
$0 = v^* B = v^* A B = v^* A^2 B = \dots$

Corollary If $\Lambda(A) \subseteq LHP$, then the solution of
$AW + WA^* + BB^* = 0$ satisfies $W \succ 0$ iff (A, Q) controllable.

Controllable means controllable

Theorem

(A, B) controllable iff for any "target" (t_F, x_F) (typycally, $x_F = 0$) we can choose a control u such that the system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0$$

has $x(t_F) = x_F$.

Proof

$\Rightarrow$ If (A, B) is not controllable, then $x(t) \in K(A, B)$ for all t.

$\Leftarrow$ Recall that (solution of linear differential eqns)

$$x(t) = \exp(At)x_0 + \int_0^t \exp(A(t-\tau))Bu(\tau)\mathrm{d}\tau.$$

Just take $u(t) = B^* \exp(A(t-\tau))^* y$ (for a fixed vector y) to get

$$x(t_F) = \exp(At_F)x_0 + Wy,$$

which can 'reach' arbitrary vectors since $W \succ 0$ is nonsingular.

Stabilizability

Weaker condition: sometimes even if a system is not controllable we can still ensure that the solution converges to 0 via a feedback control.

Definition

(A, B) is stabilizable if in its Kalman decomposition A_{22} is stable (i.e., $\Lambda(A_{22}) \subseteq LHP$).

Note that this definition is well-posed even if M is non-unique: the eigenvalues of A_{11} are the eigenvalues of $A|_{K(A,B)}$, and those of A_{22} are the remaining eigenvalues of A (counting with their algebraic multiplicity).

Hautus test: (A, B) stabilizable $\iff$ $\text{rank}(A - zI, B) = n$ for all $z \notin LHP$.

How to test controllability numerically?

Numerically, almost any (A, B) is controllable: things are rarely zero. Anyway, various options:

- Run a (block) Krylov algorithm, and check if it breaks down early.
- Compute $\Lambda(A)$ and check that $\operatorname{rank}[A - zI, B] = n$ for each $z \in \Lambda(A)$.
- If $\Lambda(A) \subset LHPs$, then you can also solve the Lyapunov equation $AW + WA^* + BB^* = 0$ and see if the solution is $\succ 0$.

What if $\Lambda(A) \not\subset LHP$? You can use the following result:

$K(A - \alpha I, B) = K(A, B)$, hence $(A - \alpha I, B)$ is controllable iff (A, B) is.

Proof For all $j \in \mathbb{N}$, $(A - \alpha I)^j B$ is a linear combination of $B, AB, A^2 B \dots$ hence $K(A - \alpha I, B) \subseteq K(A, B)$. And vice versa.

How to test controllability numerically?

Remark The criterion with the Lyapunov equation actually corresponds to a physical quantity: $x_0^* W^{-1} x_0$ is the minimal amount of energy $\int_0^{t_F} u(\tau)^* u(\tau) d\tau$ that we need to reach $x(t_F) = 0$ starting from $x(0) = x_0$. (We won't prove it here.)

So the closer (A, B) is to non-controllability, the more energy you need to 'control' certain initial states.

(Matlab examples: construct a non-controllable (A, B) from a Kalman decomposition, and apply the various methods.)

Similarly, there are an infinite number of choices for F that yield a stable $\Lambda(A + BF) \subset LHP$ (by continuity, for instance.)

- How to find one?
- How to find the best one (and what does it even mean)?

How to find a stabilizing control: Bass algorithm

Given a controllable (A, B), how can we compute F so that $\Lambda(A + BF) \subset LHP$?

Let $\alpha > \rho(A)$; then $\Lambda(-A - \alpha I) \subseteq LHP$, and the Lyapunov eq.

$$-(A + \alpha I)W - W(A + \alpha I)^* + 2BB^* = 0$$

has a solution $W \succeq 0$. It is actually $W \succ 0$, because $(-A - \alpha I, B)$ is controllable iff (A, B) is.

Some algebra gives another Lyapunov equation

$$(A - BB^*W^{-1})W + W(A - BB^*W^{-1})^* + 2\alpha W = 0.$$

Earlier result: $W \succ 0$, $2\alpha W \succ 0 \implies \Lambda(A - B(B^*W^{-1})) \subset LHP$.

Remark If (A, B) is controllable, we can find F such that $A + BF$ has any chosen spectrum. (We won't prove it here.) [Datta, Ch. 11]